STRESS-BASED FINITE ELEMENT METHOD FOR EULER-BERNOULLI BEAMS

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ABSTRACT

This paper presents a new technique, which can apply the stress-based finite element method to Euler-Bernoulli beams. An approximated bending stress distribution is selected, and then the approximated transverse displacement is determined by twice integration. Due to the satisfaction of compatibility, the integration constants are determined by the boundary conditions related to transverse displacement and rotation. To compare with the displacement-based finite element method, this technique provides the continuities of not only transverse displacement and rotation but also stress at nodes. Besides, the boundary conditions related to stress are satisfied. Two numerical examples demonstrate the validity of this technique. The results show that the errors are smaller than those generated by the displacement-based finite element method for the same number of degrees of freedom.

MÉTHODE D'ÉLÉMENT FINIE À BASE DE TENSION POUR LES BIELLES d'EULER-BERNOULLI

RÉSUMÉ

Ce papier présente une nouvelle technique, qui peut appliquer la méthode d'élément finie à base de tension aux bielles d'Euler-Bernoulli. Une distribution de tension tournante rapprochée est choisie et ensuite le déplacement transversal rapproché est déterminé par deux fois l'intégration. En raison de la satisfaction de compatibilité, l'intégration constante est déterminée par les conditions qui marquent la limite rattachées au déplacement transversal et à la rotation. Pour être comparable avec la méthode d'élément finie à base de déplacement, cette technique fournit les continuités de pas le déplacement seulement transversal et la rotation, mais insister aussi aux noeuds. En plus les conditions qui marquent la limite rattachées pour insister sont satisfaites. Deux exemples numériques démontrent la validité de cette technique. Les résultats montrent que les erreurs sont plus petites que ceux produits par la méthode d'élément finie à base de déplacement pour le même nombre de degrés de liberté.

1. INTRODUCTION

The displacement-based finite element (DFE) method employs complementary energy by imposing assumed displacements. This method may yield the discontinuities of stress fields on the inter-element boundary while employing low-order elements, and the boundary conditions associated with stress could not be satisfied. Hence, an alternative approach was developed and called the stress-based finite element (SFE) method, which utilizes assumed stress functions. This method is rarely applied to most engineering problems. The reasons are the difficulty of constructing a simple stress function and the satisfaction of stress boundary conditions in two- and three-dimensional problems.

Veubeke and Zienkiewicz [1, 2] were the first researchers introducing the SFE method. After that, the SFE method was applied to a wide range of problems, such as Kirchhoff plates [3,4], plane elastic problems [5,6], and elasto-plastic analysis [6,7]. Taylor *et al.* [8] utilized the penalty function to satisfy equilibrium equations. Gallagher *et al.* [9] used the Airy stress function to construct assumed stress functions. Vallabhan *et al.* [10] used the Lagrange multipliers to include the boundary conditions as constraints in the Lagrangian. The above works addressed two-dimensional problems using assumed stress functions, and imposed the constraint of the boundary conditions into the complementary energy. Two-dimensional problems can employ the Airy stress functions to satisfy stress boundary conditions, but Euler-Bernoulli beams do not have them to employ. Aslo, Euler-Bernoulli beams require the continuities of the first derivative of the transverse displacement. However, to the best knowledge of the authors, a review of the literature revealed that no papers implemented the SFE method on the Euler-Bernoulli beams.

This paper presents a new technique for the implementation of the SFE method to the Euler-Bernoulli beams. The developed technique first selects an assumed stress function. Then, the approximated transverse displacement function is obtained by integrating the assumed stress function. Thus, this approach can satisfy the stress boundary conditions without imposing a constraint. Illustrative numerical examples are provided to demonstrate the validity of this approach.

2. SFE METHOD FOR EULER-BERNOULLI BEAMS

The bending stress of Euler-Bernoulli beams is associated with the second derivative of the transverse displacement, namely curvature, which can be approximated as the product of shape functions and nodal variables:

$$^{(i)}\sigma = Ey\frac{d^2}{dx^2}{}^{(i)}v = Ey[^{(i)}N_{(C)}]\{^{(i)}\phi_e\} \tag{1}$$

where $[^{(i)}N_{(C)}]$ is a row vector of shape functions for the ith element ; $\{^{(i)}\phi_e\}$ is a column vector of nodal curvatures, y is the lateral position with respect to the neutral line of the beam, E is the Young's modulus, and $^{(i)}v$ is the transverse displacement, which is a function of axial position x.

Integrating Eq. (1) leads to the expressions of the rotation and the transverse displacement as

$$\text{Rotation: } \frac{d}{dx}{}^{(i)}v = \int\left[^{(i)}N_{(C)}\right]\left\{^{(i)}\phi_e\right\}dx + {}^{(i)}C_1 \tag{2}$$

$$\text{Transverse displacement: } {}^{(i)}v = \iint\left[^{(i)}N_{(C)}\right]\left\{\phi_e\right\}dxdx + {}^{(i)}C_1x + {}^{(i)}C_2 \tag{3}$$

where $^{(i)}C_1$ and $^{(i)}C_2$ are two integration constants for the ith element, which can be determined by satisfying the compatibility.

Substituting Eqs. (2) and (3) into (1), the finite element displacement, rotation and curvature can be expressed as:

$$\begin{Bmatrix} {}^{(i)}v_{xx} \\ {}^{(i)}v_{x} \\ {}^{(i)}v \end{Bmatrix} = \begin{Bmatrix} \left[{}^{(i)}N_{(C)}\right] \\ \left[{}^{(i)}N_{(R)}\right] \\ \left[{}^{(i)}N_{(D)}\right] \end{Bmatrix} \{\phi\} = \left[{}^{(i)}N\right]\{\phi\} \tag{4}$$

where the subscripts (*C*), (*R*) and (*D*) refer to curvature, rotation and displacement, respectively.

By applying the variational principle, the element and global equations can be obtained.

3. COMPARISONS OF THE DFE AND SFE METHODS

The major disadvantage of the DFE method is that the stress fields at the inter-element nodes are discontinuous while employing low-degree shape functions. This discontinuity yields one of the major concerns behind the discretization errors. In addition, it might use excessive nodal variables while formulating stiffness matrices.

The SFE method has several advantages over the DFE method. First of all, the SFE method produces fewer nodal variables (Table 1). Secondly, when employing the SFE method, the boundary conditions of bending stress can be satisfied, and the stress is continuous at the inter-element nodes. Finally, the stress is calculated directly from the solution of the global system equations. However, the only disadvantage of the SFE method is that the integration constants are different for each element.

Table 1: Comparison of the DFE and the SFE methods for an Euler-Bernoulli beam element

	DFE method	SFE method
Degree of approximated transverse displacement	Cubic	Cubic
Degree of approximated bending stress	Linear	Linear
Nodal variables	Displacement and rotation at both ends	Curvature at both ends
Boundary conditions satisfied	Displacement, rotation	Displacement, rotation, bending stress
Number of degrees of freedom	Four	Two

4. ILLUSTRATIVE EXAMPLES

The SFE and DFE methods are employed to two examples, and the results demonstrate the error comparison.

4.1 A cantilever beam subjected a uniformly distributed load

A cantilever beam is subjected to a uniformly distributed load. The beam parameters are given as: L (beam's length) = 10 (m), q (density of distributed load) = 100 (N/m), $EI = 1.4 \times 10^4$ (N-m^2), and ρA (product of mass per unit volume and cross-section area) = 1.2 (kg/m).

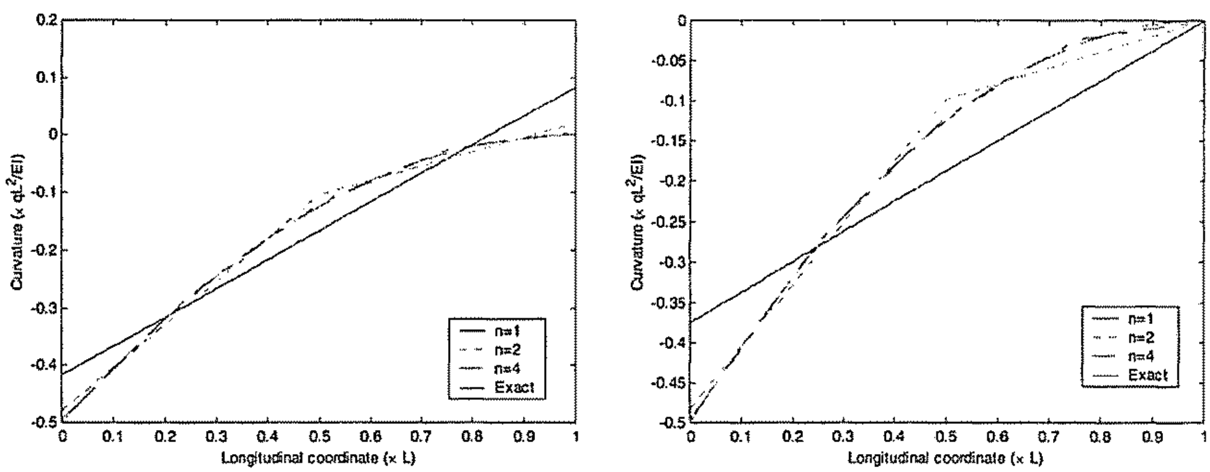
Figs. 1 shows the curvature distribution of the beam by the DFE and the SFE methods. Table 2 lists error comparisons for both methods. When examining the errors of the strain energy for the same number of elements, the errors obtained by employing the DFE method are slightly smaller than those from the

SFE method. When considering the same number of degrees of freedom, the errors from the SFE method are much smaller than those from the DFE method.

Table 2: Errors of the DFE and SFE methods for a cantilever beam

Number of elements	Error of strain energy by		Error of clamped-end stress by	
	DFE method	SFE method	DFE method	SFE method
1	0.1667 (DOF=2)	0.2500 (DOF=1)	0.1667 (DOF=2)	0.2500 (DOF=1)
2	4.167E-2 (DOF=4)	5.455E-2 (DOF=2)	4.167E-2 (DOF=4)	3.571E-2 (DOF=2)
3	1.852E-2 (DOF=6)	2.253E-2 (DOF=3)	1.852E-2 (DOF=6)	1.923E-2 (DOF=3)
4	1.042E-2 (DOF=8)	1.215E-2 (DOF=4)	1.042E-2 (DOF=8)	1.031E-2 (DOF=4)

DOF: Degrees of freedom



(a) (b)

Figure 1: Curvature distribution by (a) the DFE and (b) the SFE methods

4.2 Slewing beam

A slewing beam is shown in Fig. 2, and there is a rigid body rotational angle $\theta(t)$. Mathematical models are established with respect to the moving coordinate system. This beam is assumed to have small elastic transverse displacement v, and it is inextensible in its longitudinal direction. Fig. 3 illustrates a beam element with the translating and rotating motions.

The Lagrangian for a translating and rotating beam element can be expressed as

$$L_e = T_e - U_e - Y_e \tag{5}$$

where T_e, U_e and Y_e are the kinetic energy, the flexural strain energy, and the work done by a tensile longitudinal load P_e, respectively [11]. Using Lagrange's equation and assembling all elements, the discrete governing equations of motion can be expressed as:

$$[M]\{\ddot{\phi}\}+[C]\{\dot{\phi}\}+[K]\{\phi\}=\{F\} \tag{6}$$

where $[M]$, $[C]$, $[K]$ are global mass, equivalent damping and stiffness matrices, respectively; $\{F\}$ is a global load column vector.

A prescribed rotation angle $\theta(t)$ applied at the slewing beam is considered as follows:

$$\theta(t)=\begin{cases}\frac{\dot{\theta}_{ss}}{t_{ss}}[\frac{t^2}{2}+(\frac{t_{ss}}{2\pi})^2(\cos\frac{2\pi t}{t_{ss}}-1)] \quad (\text{rad}), & 0\le t\le t_{ss}\\ \dot{\theta}_{ss}(t-\frac{t_{ss}}{2}) \quad (\text{rad}), & t\ge t_{ss}\end{cases} \tag{7}$$

where the rotating speed maintains at $\dot{\theta}_{ss}$ after the time t_{ss}. In this example, parameters $\dot{\theta}_{ss}$ and t_{ss} are selected as $\dot{\theta}_{ss}=6$ (rad/s) and $t_{ss}=15$ (sec) [12].

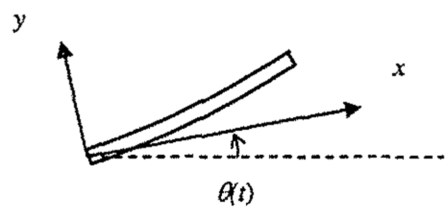


Figure 2: A slewing beam

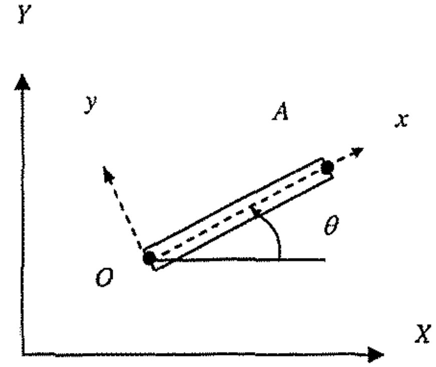


Figure 3: A rotating and translating beam element

Tables 3 and 4 list the various errors from the DFE and SFE methods, respectively [13]. Both approximated displacements are cubic polynomials. When considering the same number of elements, the errors obtained from the SFE method are larger than those from the DFE method. However, when examining the same number of degrees of freedom, the errors obtained from the SFE method are smaller.

Table 3: Errors obtained by the SFE for a slewing beam

Number of elements	Degrees of freedom	Errors of			
		Total energy	Tip deflection	Tip rotation	Clamped-end stress
1	1	5.333E-4	2.486E-2	1.059E-1	2.956E-1
2	2	1.377E-4	4.538E-3	1.658E-2	8.231E-2
3	3	3.337E-5	1.021E-3	5.744E-3	4.522E-2
4	4	1.054E-5	3.241E-4	2.481E-3	2.665E-2

5. CONCLUSIONS

This paper presents a new technique of applying the SFE method to Euler-Bernoulli beams, and demonstrates its validity by numerical examples. An approximated curvature distribution is selected first, and then an approximated transverse displacement is determined by integration. The approximated bending stress is continuous at inter-element nodes, and the boundary conditions of bending stress can be satisfied. To compare with the DFE method, the results show that the errors from the SFE method are smaller when considering the same number of degrees of freedom.

Table 4: Errors obtained by the DFE method for a slewing beam

Number of elements	Degrees of freedom	Errors of			
		Total energy	Tip deflection	Tip rotation	Clamped-end stress
1	2	5.616E-4	2.126E-2	2.586E-2	2.135E-1
2	4	1.050E-4	3.509E-3	3.377E-3	7.698E-2
3	6	2.555E-5	8.347E-4	7.820E-4	3.998E-2
4	8	8.710E-6	2.823E-4	1.624E-4	2.442E-2

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