STABILITY ANALYSIS OF THE THIN POWER LAW LIQUID FILM FLOWING DOWN ON A VERTICAL CYLINDER

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ABSTRACT

The paper investigates the stability theory of a thin power law liquid film flowing down along the outside surface of a vertical cylinder. The long-wave perturbation method is employed to solve for generalized linear kinematic equations with free film interface. The normal mode approach is used to compute the stability solution for the film flow. The degree of instability in the film flow is further intensified by the lateral curvature of cylinder. This is somewhat different from that of the planar flow. The analysis results also indicate that by increasing the flow index and increasing the radius of the cylinder the film flow can become relatively more stable as traveling down along the vertical cylinder.

ANALYSE DE LA STABILITÉ D'UNE PELLICULE MINCE LIQUIDE DE LOI DE PUISSANCE QUI S'ÉCOULE LE LONG D'UN CYLINDRE VERTICAL

RÉSUMÉ

Cet article examine la théorie de la stabilité d'une pellicule mince liquide de loi de puissance qui s'écoule le long de la surface d'un cylindre vertical. La méthode de perturbation à ondes longues est employée pour résoudre les équations cinématiques linéaires généralisées avec une interface de pellicule libre. L'approche de mode normal est utilisée en vue de déterminer la solution de stabilité pour le débit de la pellicule. Le degré d'instabilité du débit de la pellicule est davantage intensifié par la courbure latérale du cylindre, ce qui est plutôt différent du débit planaire. Les résultats de l'analyse indiquent également que, en augmentant l'indice de débit et le radius du cylindre, le débit de la pellicule devient relativement plus stable à mesure qu'elle s'écoule le long du cylindre vertical.

1. INTRODUCTION

The stability of a film flow is a research subject of great importance commonly needed in mechanical, chemical and nuclear engineering industries for various applications including the process of paint finishing, the process of laser cutting and heavy casting production processes. It is known that macroscopic instabilities can cause disastrous conditions to fluid flow. It is thus highly desirable to understand the underlying flow characteristics and associated time-dependent properties so that suitable conditions for homogeneous film growth can be developed for various industrial applications.

The problem of the stability of the laminar flow of an ordinary viscous liquid film flowing down an inclined plane under gravity was first formulated and solved numerically by Yih [1]. The transition mechanism from laminar flow to turbulent flow was elegantly explained by the Landau equation [2]. That shed light for later development on nonlinear film stability. The Landau equation was later re-derived by Stuart [3] using the disturbed energy balance equation along with Reynolds stresses. Benjamin [4] and Yih [5] formulated the disturbed wave equation of free flow surface. The flow stability of long disturbed wave was carefully studied and some characteristics of the flow stability on an inclined plane are observed. Benney [6] investigated the nonlinear evolution equation of free surface by using the method of small parameters. The solutions thus obtained can be used to predict nonlinear instability. The effect of surface tension was realized by many researchers as one of the necessary conditions that will lead to the solution of supercritical stability. Lin [7], Nakaya [8] and Krishna and Lin [9] considered the significance of surface tension and treated it in terms of first order terms in later studies. Pumir et al. [10] further included the effect of surface tension into the film flow model and solved for the solitary wave solutions. In order to fully understand and characterize the stability conditions for various film flows, detailed flow analysis is of great importance.

Several researchers have already studied the hydrodynamic stability problems regarding the fluid films flowing down a vertical cylinder surface. Lin and Liu [11] compared their analytical solutions with the existing experimental results of falling flow film on a cylinder and creeping annular flow threads in viscous liquid. Krantz and Zollars [12] presented an asymptotic solution and pointed out that the effect of curvature on the stability of the film flow is indeed significant. They also showed that the curvature of the cylinder is indeed one of the important factors that intensify the instability of the film flow. This phenomenon is not found in the planar flow. Rosenau and Oron [13] derived an amplitude equation which describes the evolution of a disturbed free film surface traveling down an infinite vertical cylindrical column. The numerical analysis results indicated that both conditions of supercritical stability and subcritical instability are possible to occur for the film flow. The results also showed that the evolving waves may break at the instant that linearly unstable conditions are satisfied. Davalos-Orozco and Ruiz-Chavarria [14] investigated the linear stability of a fluid layer flowing down inside and outside of a rotating vertical cylinder. They pointed out that the centrifugal force could stabilize the film flow so as to counteract the destabilizing effect of surface tension. In the absence of rotation, the stability can still be found for some critical wave numbers. Hung et al. [15] investigated the weakly nonlinear stability analysis of a condensation film flowing down a vertical cylinder. They also showed that supercritical stability in the linearly unstable region and subcritical instability in the linearly stable region can co-exist. They also indicated that the lateral curvature of the cylinder has the destabilizing effect on the film flow stability. Shlang and Sivashinsky [16] investigated the irregular flow of a liquid film down a vertical column. Cheng and Chang [17] considered the stability of azimuthal and streamwise disturbances on a layer of viscous fluid flowing down a cylindrical surface. Their results revealed that both streamwise and transverse modulations are responsible for instability of a viscous fluid flowing down a cylindrical surface. They also claimed that the stability of thin film flow is more susceptible to azimuthal disturbances as the column radius increases.

A vast majority of studies on thin-film flow problems were devoted to the stability analysis of Newtonian fluids. The film flow of non-Newtonian fluids attracted less attention in the past. The rheological behaviors of fluids during the plastic manufacture, the lubrication of bearings or the glue in biological chemistry do not obey the Newtonian postulate. In recent years, the microstructure of fluid flows has emerged as a research subject of great interest to many researchers. Cheng et al. [18] employed the method of nonlinear analysis to study the nonlinear stability of thin micropolar liquid film flowing down on a vertical cylinder. The results of their study indicated that the micropolar parameter plays an important role in stabilizing a film flow. The viscoelastic fluid, a subclass of microstructure flows, exhibits a great deal of influence on the normal and shear stresses in flow films. The stability problem of a falling film of viscoelastic fluid has been studied by Gupta [19] who considered the stability of a small-amplitude falling fluid of second order. The long wavelength disturbance is used in the paper to conduct a linear stability analysis. After deriving the viscoelastic analog of the Orr-Sommerfeld equation with the requisite boundary condition, Gupta pointed out the viscoelastic effect can destabilize the film flow. Cheng et al. [20] further studied the nonlinear stability analysis of thin viscoelastic liquid film flowing down on a vertical wall cylinder. They also demonstrated that the viscoelastic property has destabilizing effect on the nonlinear film flow system. Hwang et al. [21] studied the linear stability of power law liquid film flows down an inclined plane by using the integral method. The results reveal that the system will be more unstable when power-law exponent n decreases.

In practical applications, pseudoplastic fluids $(n<1)$ that show shear thinning and dilantant fluids $(n>1)$ that show shear thickening are widely used in the analysis to characterize the various fluids. The stability analysis of a power-law liquid film flow has been studied by several authors [22, 23, 24]. Miladinova et al. [23] demonstrated that the maximum wave amplitude is always smaller in the case of a shear-thickening liquids $(n>1)$ than in case of a Newtonian liquid. Gorla [24] displayed the rupture times for the dilatant fluids $(n > 1)$ are higher than that of Newtonian fluid.

The stability analysis of the power law liquid film flow is indeed an interesting research area in both theoretical development and practical applications. To our best knowledge the stability analysis of a thin power law liquid film flowing down a vertical cylinder has not been seriously investigated so far. However, since the types of stability problems are of great importance in many practical applications, the behavior of a power law liquid film traveling down along a vertical cylinder is carefully studied in this paper by employing stability analysis theories. The influence of both the flow index and the cylinder size on finite-amplitude equilibrium is studied and characterized mathematically. The sensitivity analysis of both the power law and cylinder size is also carefully conducted. Several numerical examples are presented to verify the solutions and to demonstrate the effectiveness of the proposed modeling procedure.

2. SIMULATION MODELS

Figure 1 shows the configuration of a thin power law liquid film flowing down along the outer surface of an infinite vertical cylinder. All physical properties are assumed to be constant. The principles of mass and momentum conservation for an axisymmetric isothermal incompressible power law flow configuration leads one to a set of system governing equations. Let u^* and w^* be the velocity components in r^* and z^* directions, respectively.

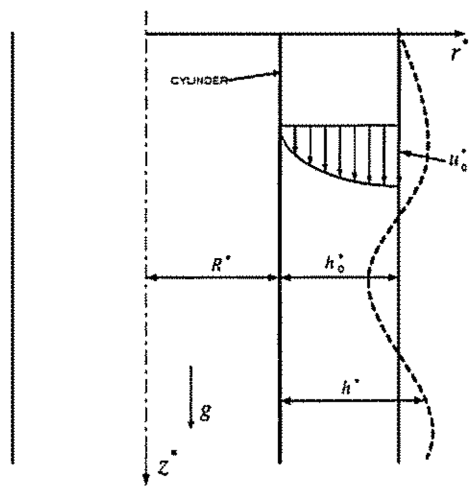


Fig. 1 Schematic diagram of a power law thin film flow traveling down along a vertical cylinder

The governing equations can be expressed in terms of cylindrical coordinates $(r^*,\ z^*)$ as

$$\frac{1}{r^*}\frac{\partial(r^* u^*)}{\partial r^*}+\frac{\partial w^*}{\partial z^*}=0 \tag{1}$$

$$\rho\left(\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial r^*}+w^*\frac{\partial u^*}{\partial z^*}\right)=\frac{1}{r^*}\frac{\partial(r^*\tau_{r^*r^*})}{\partial r^*}+\frac{\partial\tau_{z^*r^*}}{\partial z^*}-\frac{1}{r^*}\tau_{\theta^*\theta^*} \tag{2}$$

$$\rho(\frac{\partial w^*}{\partial t^*}+u^*\frac{\partial w^*}{\partial r^*}+w^*\frac{\partial w^*}{\partial z^*})=\frac{1}{r^*}\frac{\partial(r^*\tau_{r^*z^*})}{\partial r^*}+\frac{\partial\tau_{z^*z^*}}{\partial z^*}+\rho g \tag{3}$$

where ρ is a constant density of the film flow, t^* is the time, g is the gravitational acceleration, and the individual stress components are given as

$$\tau_{r^*r^*}=-p^*+2\mu(\frac{\partial u^*}{\partial r^*})^n \tag{4}$$

$$\tau_{z^*z^*}=-p^*+2\mu(\frac{\partial w^*}{\partial z^*})^n \tag{5}$$

$$\tau_{r^*z^*}=\tau_{z^*r^*}=\mu(\frac{\partial u^*}{\partial z^*}+\frac{\partial w^*}{\partial r^*})^n \tag{6}$$

$$\tau_{\theta^*\theta^*}=-p^*+2\mu(\frac{u^*}{r^*})^n \tag{7}$$

Where n is the flow index of the film flow, μ is the fluid dynamic viscosity, and the no-slip boundary conditions on the outer wall of the cylinder at $r^*=R^*$ are given as

$$u^*=0 \tag{8}$$

$$w^*=0 \tag{9}$$

The boundary conditions at free surface of $r^*=R^*+h^*$ are derived based on the results given by Edwards et al. [25]. The vanishing of shear stress on free surface gives another boundary condition as

$$\frac{\partial h^*}{\partial z^*}[1+(\frac{\partial h^*}{\partial z^*})^2]^{-1}(\tau_{r^*r^*}-\tau_{z^*z^*})+[1-(\frac{\partial h^*}{\partial z^*})^2][1+(\frac{\partial h^*}{\partial z^*})^2]^{-1}\tau_{r^*z^*}=0 \tag{10}$$

By solving the balance equation in the direction normal to the free surface, the resulting normal stress condition can be expressed as

$$[1+(\frac{\partial h^*}{\partial z^*})^2]^{-1}[2\tau_{r^*z^*}\frac{\partial h^*}{\partial z^*}-\tau_{r^*r^*}-\tau_{z^*z^*}(\frac{\partial h^*}{\partial z^*})^2]+S_n^*\{\frac{\partial^2 h^*}{\partial z^{*2}}[1+(\frac{\partial h^*}{\partial z^*})^2]^{-3/2}-\frac{1}{r}[1+(\frac{\partial h^*}{\partial z^*})^2]^{-1/2}\}=p_a^* \tag{11}$$

The kinematic condition that the flow cannot travel across a free surface can be described as

$$\frac{\partial h^*}{\partial t^*}+\frac{\partial h^*}{\partial z^*}w^*-u^*=0 \tag{12}$$

where h^* is the local film thickness, p_a^* is the atmosphere pressure, and S_n^* is the surface tension. The variable that is associated with a superscript "*" stands for a dimensional quantity. By introducing the stream function, φ^*, into dimensional velocity components, they become

$$u^*=\frac{1}{r^*}\frac{\partial\varphi^*}{\partial z^*},\quad w^*=-\frac{1}{r^*}\frac{\partial\varphi^*}{\partial r^*} \tag{13}$$

The dimensionless quantities can also be defined and given as

$$z=\frac{\alpha z^*}{h_0^*},\quad r=\frac{r^*}{h_0^*},\quad t=\frac{\alpha u_0^* t^*}{h_0^*},\quad h=\frac{h^*}{h_0^*},\quad \varphi=\frac{\varphi^*}{u_0^* h_0^{*2}},\quad p=\frac{p^*-p_a^*}{\rho u_0^{*2}},$$

$$\mathrm{Re}_n=\frac{u_0^{*2-n}h_0^{*n}}{\nu},\quad S_n=\frac{S_n^*}{(2^{-3n^2+3n+2}\rho^{n+2}\nu^4 g^{3n-2})^{\frac{1}{n+2}}},\quad \alpha=\frac{2\pi h_0^*}{\lambda},\quad R=\frac{R^*}{h_0^*} \tag{14}$$

where Re_n is the Reynolds number, R is the dimensionless radius of the cylinder, λ is the perturbed wave length, ν is the fluid kinematic viscosity, and α is the dimensionless wave number.

h_0^* is the film thickness of local base flow and u_0^* is the reference velocity which can be expressed as

$$u_0^* = \frac{1}{4}(\frac{g}{\nu_0})^m \frac{h_0^{*1+m}}{\Gamma} \tag{15}$$

where

$$m = 1/n \tag{16}$$

$$\Gamma = [2(1+R)^2 \ln(\frac{1+R}{R}) - (1+2R)]^{-1} \quad (n=1) \tag{17}$$

$$\Gamma = \frac{2^{-2+m}(-2+m)(-1+m)R^m(1+R)^m(1+2R)^{1-m}}{-R(1+R)^m\{2+4R+m[-1+2(-1+R)R]\}+R^m(1+R)\{2+4R+m[-1+2R(-2+m+R)]\}} \quad (n \neq 1) \tag{18}$$

Thus, the non-dimensional governing equations and the associated boundary conditions can now be given as

$$\begin{aligned}\frac{\partial p}{\partial r} = \alpha \cdot \mathrm{Re}^{-1}\{n[2(\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi}{\partial z}))^{n-1}\frac{\partial^2}{\partial r^2}(\frac{1}{r}\frac{\partial \varphi}{\partial z}) - (\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi}{\partial r}))^{n-1}\frac{\partial^2}{\partial r \partial z}(\frac{1}{r}\frac{\partial \varphi}{\partial r})] + \\ \frac{2}{r}[(\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi}{\partial z}))^n - (\frac{1}{r^2}\frac{\partial \varphi}{\partial z})^n]\} + O(\alpha^2)\end{aligned} \tag{19}$$

$$\begin{aligned}n[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi}{\partial r})]^{n-1}\frac{\partial^2}{\partial r^2}(\frac{1}{r}\frac{\partial \varphi}{\partial r}) + \frac{1}{r}[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi}{\partial r})]^n = (4\Gamma)^n + \alpha \mathrm{Re}[-\frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial^2 \varphi}{\partial t \partial r} \\ + \frac{1}{r^2}\frac{\partial \varphi}{\partial z}\frac{\partial^2 \varphi}{\partial r^2} - \frac{1}{r^3}\frac{\partial \varphi}{\partial z}\frac{\partial \varphi}{\partial r} - \frac{1}{r^2}\frac{\partial \varphi}{\partial r}\frac{\partial^2 \varphi}{\partial r \partial z}] + O(\alpha^2)\end{aligned} \tag{20}$$

at the cylinder surface $(r = R)$

$$\varphi = \frac{\partial \varphi}{\partial r} = \frac{\partial \varphi}{\partial z} = 0 \tag{21}$$

at free surface $(r = R + h)$

$$\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi}{\partial r}) = 0 + O(\alpha^2) \tag{22}$$

$$p = -2S_n \cdot \mathrm{Re}^{\frac{-3n^2+4n+4}{(n+2)(n-2)}}(2\Gamma)^{\frac{(3n-2)n}{n+2}}(\alpha^2\frac{\partial^2 h}{\partial z^2} - \frac{1}{r}) + \alpha\frac{2}{\mathrm{Re}}[(\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi}{\partial r}))^n\frac{\partial h}{\partial z} + (\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi}{\partial z}))^n] + O(\alpha^2) \tag{23}$$

$$\frac{\partial h}{\partial t} - \frac{1}{r}(\frac{\partial \varphi}{\partial r}\frac{\partial h}{\partial z} + \frac{\partial \varphi}{\partial z}) \tag{24}$$

3. STABILITY ANALYSIS

Since the long wave length modes (i.e., small wave number α) may introduce flow instability to meet our analysis objectives, the dimensionless stream function φ and pressure p are, therefore, expanded here in terms of some small wave number α as

$$\varphi = \varphi_0 + \alpha\varphi_1 + O(\alpha^2) \tag{25}$$

$$p = p_0 + \alpha p_1 + O(\alpha^2) \tag{26}$$

By plugging the above two equations into Eqs. (19)-(24), the system governing equations can then be collected and solved order by order. In practice, the non-dimensional surface tension S_n is a large value, the term $\alpha^2 S_n$ can be treated as a quantity of zeroth order [15, 18]. After collecting all terms of zeroth order (α^0) from the governing equations, a set of zeroth order equations is obtained and presented as

$$n[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi_0}{\partial r})]^{n-1}\frac{\partial^2}{\partial r^2}(\frac{1}{r}\frac{\partial \varphi_0}{\partial r})+\frac{1}{r}[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi_0}{\partial r})]^n=(4\Gamma)^n \tag{27}$$

$$\frac{\partial p_0}{\partial r}=0 \tag{28}$$

The boundary conditions associated with the equations of zeroth order are given as

$$\varphi_0=\frac{\partial \varphi_0}{\partial r}=\frac{\partial \varphi_0}{\partial z}=0, \quad \text{at} \quad r=R \tag{29}$$

$$\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi_0}{\partial r})=0, \quad \text{at} \quad r=R+h \tag{30}$$

$$p_0=-2S\cdot \mathrm{Re}^{\frac{-3n^2+4n+4}{(n+2)(n-2)}}(2\Gamma)^{\frac{(3n-2)n}{n+2}}(\alpha^2\frac{\partial^2 h}{\partial z^2}-\frac{1}{q}), \quad \text{at} \quad r=R+h \tag{31}$$

The solution at zeroth order for stream function is

$$\varphi_0=\frac{\Gamma^n}{16n}\{-2^{1+2n}[r^2R^2+(-r^2+R^2)q^2]+4^n(r^4+R^4+4r^2q^2\ln\frac{R}{r})\} \tag{32}$$

where

$$q=R+h \tag{33}$$

After collecting all terms of the first order (α^1) from the governing equations, a set of the first-order equations is given as

$$n[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi_1}{\partial r})]^{n-1}\frac{\partial^2}{\partial r^2}(\frac{1}{r}\frac{\partial \varphi_1}{\partial r})+\frac{1}{r}[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi_1}{\partial r})]^n=\mathrm{Re}[-\frac{\partial p_0}{\partial z}+\frac{1}{r}\frac{\partial^2 \varphi_0}{\partial t\partial r}+\frac{1}{r^2}\frac{\partial \varphi_0}{\partial z}\frac{\partial^2 \varphi_0}{\partial r^2}$$
$$-\frac{1}{r^3}\frac{\partial \varphi_0}{\partial z}\frac{\partial \varphi_0}{\partial r}-\frac{1}{r^2}\frac{\partial \varphi_0}{\partial r}\frac{\partial^2 \varphi_0}{\partial r\partial z}] \tag{34}$$

$$\frac{\partial p_1}{\partial r}=\mathrm{Re}^{-1}\{n[2(\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi_0}{\partial z}))^{n-1}\frac{\partial^2}{\partial r^2}(\frac{1}{r}\frac{\partial \varphi_0}{\partial z})-(\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi_0}{\partial r}))^{n-1}\frac{\partial^2}{\partial r\partial z}(\frac{1}{r}\frac{\partial \varphi_0}{\partial r})]+$$
$$\frac{2}{r}[(\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi_0}{\partial z}))^n-(\frac{1}{r^2}\frac{\partial \varphi_0}{\partial z})^n]\} \tag{35}$$

The boundary conditions associated with the equations of first order are given as

$$\varphi_1=\frac{\partial \varphi_1}{\partial r}=\frac{\partial \varphi_1}{\partial z}=0, \quad \text{at} \quad r=R \tag{36}$$

$$\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi_1}{\partial r})=0, \text{ at } \quad r=R+h \tag{37}$$

$$p_1=\frac{2}{\mathrm{Re}}\{[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi_0}{\partial r})]^n\frac{\partial h}{\partial z}+[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \varphi_0}{\partial z})]^n\}, \quad \text{at} \quad r=R+h \tag{38}$$

The solutions obtained by solving the equations of first order are given as

$$\varphi_1=\frac{\mathrm{Re}}{768n^3q^2}\{\Gamma^{2n}q^3[16^n(r-R)(r+R)(r^4-20r^2R^2+13R^4)+32^{2+4n}r^4R^2\ln\frac{r}{R}-32^{1+4n}q^2(7r^4$$
$$-8r^2R^2+R^4+4r^2(r-R)(r+R)(\ln r)^2-2\ln R(r^4-14r^2R^2+7R^4+2r^2(r^2-3R^2)\ln R)$$
$$+2(5r^4-12r^2R^2+7R^4+4r^2(r^2-2R^2)\ln R)\ln q+4r^2\ln r(2r^2+R^2+2R^2\ln R+2(r^2$$
$$-2R^2)\ln q)+(-r^2+R^2+2r^2\ln\frac{r}{R})(-3+4\ln R+8(\ln R)^2-4(1+4\ln R)\ln q+8(\ln q)^2)q^2)]h_z$$
$$+96n^2S_n\,\mathrm{Re}^{\frac{-3n^2+4n+4}{(n+2)(n-2)}}(2\Gamma)^{\frac{(3n-2)n}{n+2}}((r^2-R^2)^2-2(-r^2+R^2+2r^2\ln\frac{r}{R})q^2)(h_z+\alpha^2q^2h_{zzz})\} \tag{39}$$

By plugging the solutions for the equations of both the zeroth and the first orders into the dimensionless free surface kinematic equation of Eqn. (24), the generalized nonlinear kinematic equation is obtained and presented as

$$h_t + A(h)h_z + B(h)h_{zz} + C(h)h_{zzzz} + D(h)h_z^2 + E(h)h_z h_{zzz} = 0 \tag{40}$$

where

$$A(h) = \frac{2^{-1+2n}\Gamma^n}{n}(-q^2 + R^2 + 2q^2 \ln Q) \tag{41}$$

$$\begin{aligned} B(h) = \frac{\mathrm{Re}\,\alpha}{768 n^3 q^3}\{ & 32^{-3+3n+\frac{16}{2n}} n^2 \mathrm{Re}^{-3+\frac{4}{2+n}} S_n \Gamma^{\frac{n(-2+3n)}{2+n}} (-3q^4 + 4q^2R^2 - R^4 + 4q^4 \ln Q) \\ & + q^3\Gamma^{2n}[-32^{1+4n} q^2R^2(11q^2 - R^2) + 16^n(59q^6 + 21q^4R^2 - 33q^2R^4 + 13R^6) \\ & + 34^{1+2n} q^2(-17q^2R^2 + 7R^4 + 2q^2(-5h(q+R) + 4q^2 \ln Q)\ln Q)\ln Q]\} \end{aligned} \tag{42}$$

$$C(h) = \frac{2^{-11+3n+\frac{16}{2+n}}}{nq} \mathrm{Re}^{-\frac{2n}{2+n}} S_n \alpha^3 \Gamma^{\frac{n(-2+3n)}{2+n}} (-3q^4 + 4q^2R^2 - R^4 + 4q^4 \ln Q) \tag{43}$$

$$\begin{aligned} D(h) = \frac{\mathrm{Re}\,\alpha}{768 n^3 q^4}\{ & 32^{-3+3n+\frac{16}{2n}} n^2 \mathrm{Re}^{-3+\frac{4}{2+n}} S_n \Gamma^{\frac{n(-2+3n)}{2+n}} (-q^4 + R^4 + 4q^4 \ln Q) \\ & + q^3\Gamma^{2n}[16^n(413q^6 - 429q^4R^2 + 3q^2R^4 + 13R^6) + 12q^2(16^n R^2(-65q^2 + 21R^2) \\ & + 2^{1+4n} q^2(-10q^2 + (-23q^2 + 25R^2 + 28q^2 \ln Q)\ln Q))\ln Q]\} \end{aligned} \tag{44}$$

$$E(h) = \frac{2^{\frac{n(-2+3n)}{2+n}}}{n} \mathrm{Re}^{-\frac{2n}{2+n}} S_n \alpha^3 \Gamma^{\frac{n(-2+3n)}{2+n}} (-q^2 + R^2 + 2q^2 \ln Q) \tag{45}$$

where

$$Q = \frac{R+h}{R} = \frac{q}{R} \tag{46}$$

In case of n=1.00, the fluid flow becomes a typical classical Newtonian film flow. In case of $R = \infty$, the result agrees exactly with the solution of plane flow.

The variation of film thickness in the base flow is found very small, so it is reasonable to assume that the local dimensionless film thickness equals to one. The dimensionless film thickness when expressed in perturbed state can be expressed as

$$h(t,z) = 1 + \eta(t,z), \qquad \eta = O(\alpha) \tag{47}$$

where η is a perturbed quantity to the stationary film thickness. By inserting the above equation into Eq. (40) and collecting all terms up to the order of η^3, the evolution equation of η is obtained and given as

$$\begin{aligned} & \eta_t + A\eta_z + B\eta_{zz} + C\eta_{zzzz} \\ & = -[(A'\eta + \frac{A''}{2}\eta^2)\eta_z + (B'\eta + \frac{B''}{2}\eta^2)\eta_{zz} + (C'\eta + \frac{C''}{2}\eta^2)\eta_{zzzz} + (D + D'\eta)\eta_z^2 + (E + E'\eta)\eta_z\eta_{zzz}] + O(\eta^4) \end{aligned} \tag{48}$$

The values of A, B, C, D, E and their derivatives are all evaluated at the dimensionless height, h=1, of the film flow.

If the perturbation is small, the nonlinear terms in Eq. (48) are neglected, the perturbation equation is obtained and given as

$$\eta_t + A\eta_z + B\eta_{zz} + C\eta_{zzzz} = 0 \tag{49}$$

In order to use the normal mode method for analysis, we assume that

$$\eta = a\exp[i(z - dt)] + c.c. \tag{50}$$

where a is the perturbation amplitude, and c.c. is the complex conjugate counterpart. The complex wave celerity, d, is given as

$$d = d_r + id_i = A + i(B - C) \tag{51}$$

where d_r is the wave speed, and d_i is the growth rate of the amplitudes. For $d_i > 0$, the flow is in unstable supercritical condition. For $d_i < 0$, the flow is in stable subcritical condition.

4. NUMERICAL EXAMPLES

A numerical example is presented here to illustrate the effectiveness of the proposed modeling procedure in dealing with the problem of a thin power law fluid film flowing down along a vertical cylinder. In order to validate the result of analytical derivation, a finite-amplitude perturbation generator is employed to disturb the system for stability analyses. Based on the analysis results, the condition for thin-film flow stability can now be expressed as a function of Reynolds number, Re_1, dimensionless perturbation wave number, α, dimensionless radius of cylinder, R and flow index, n. Some important conclusions are made. The analysis results are also used to compare with the analytical solutions given in this paper and some other conclusive results which appeared in the literature.

Figure 1 shows the schematic diagram of a power law liquid film traveling down along a vertical cylinder. Physical parameters that are selected for study include (1) Reynolds numbers Re_1 ranging from 0 to 10, (2) the dimensionless perturbation wave numbers α ranging from 0 to 0.12, (3) the values of flow index n including 0.95, 1.0 and 1.05, and (4) the values of dimensionless radius distance R including 10, 20, 50 and ∞. The neutral stability curve was obtained by computing the conditions of stability for an amplitude growth rate $d_i = 0$. The stability of flow field (α - Re_1 plane) is separated into two different regions by the neutral curve. In the stable sub-critical region, the perturbed small waves decay as the perturbation time period increases. However, in the unstable supercritical region, the perturbed small waves grow as the perturbation time period increases. In order to study the influence of the flow index and radius of the cylinder on the stability of the film flow, a constant dimensionless surface tension (S_1 = 6173.5) is used throughout for all numerical computations [15].

Figure 2(a)-2(b) shows the neutral stability curves of the power law film flow with different values on the flow index, n. The neutral stability curve of the Newtonian flow (i.e. n=1.00) is also given in the figure for comparison purpose. The results indicate that the area of the unstable region ($d_i > 0$) becomes larger for a decreasing n. Figure 2(c)-2(d) shows the neutral stability curves of the power law film flow with different values on the radius R. The neutral stability curve of the plane flow (i.e. R $=\infty$) is also given in the figure for comparison purpose. The results indicate that the area of the unstable region $(d_i > 0)$ becomes larger for a decreasing R.

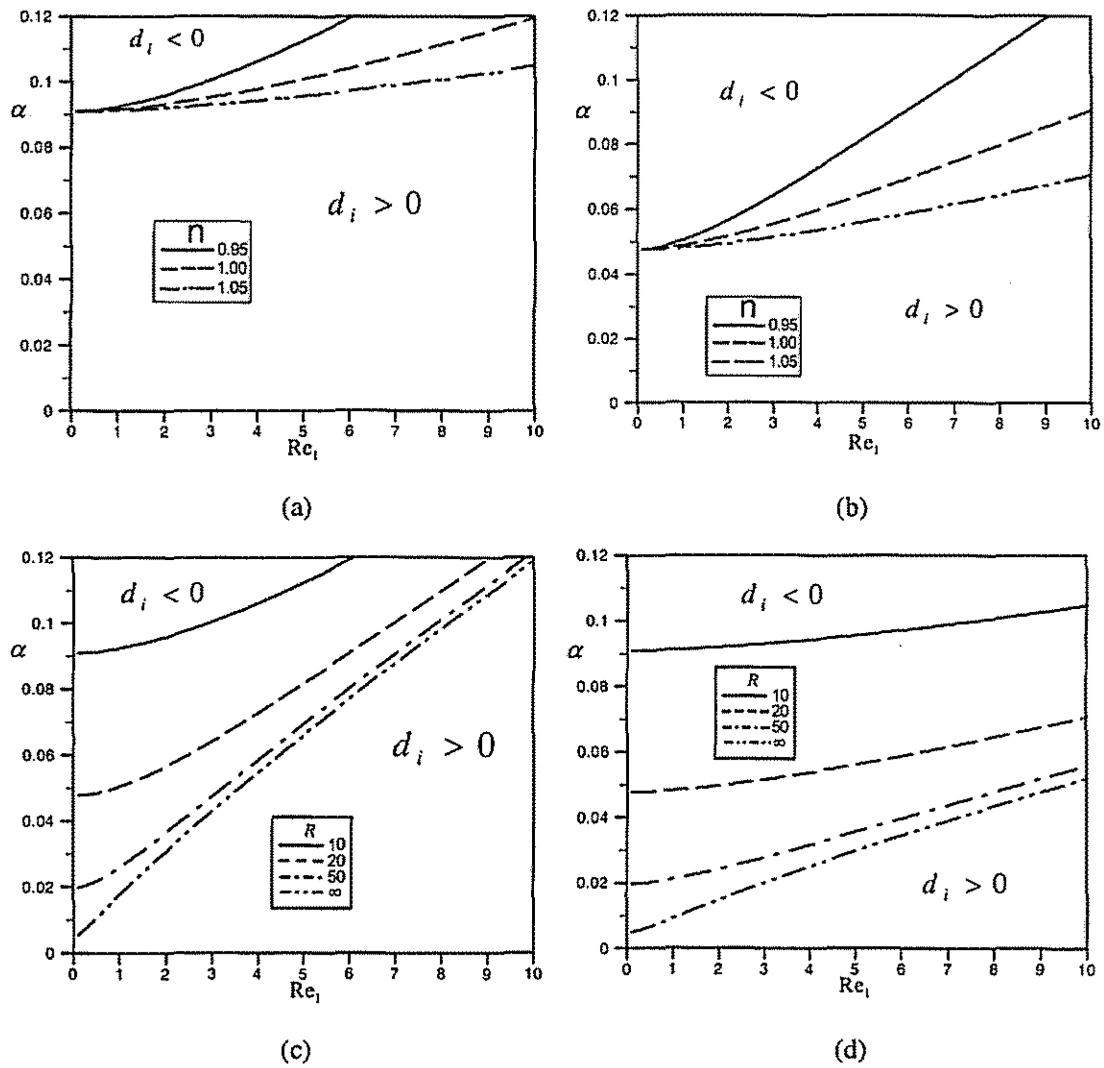
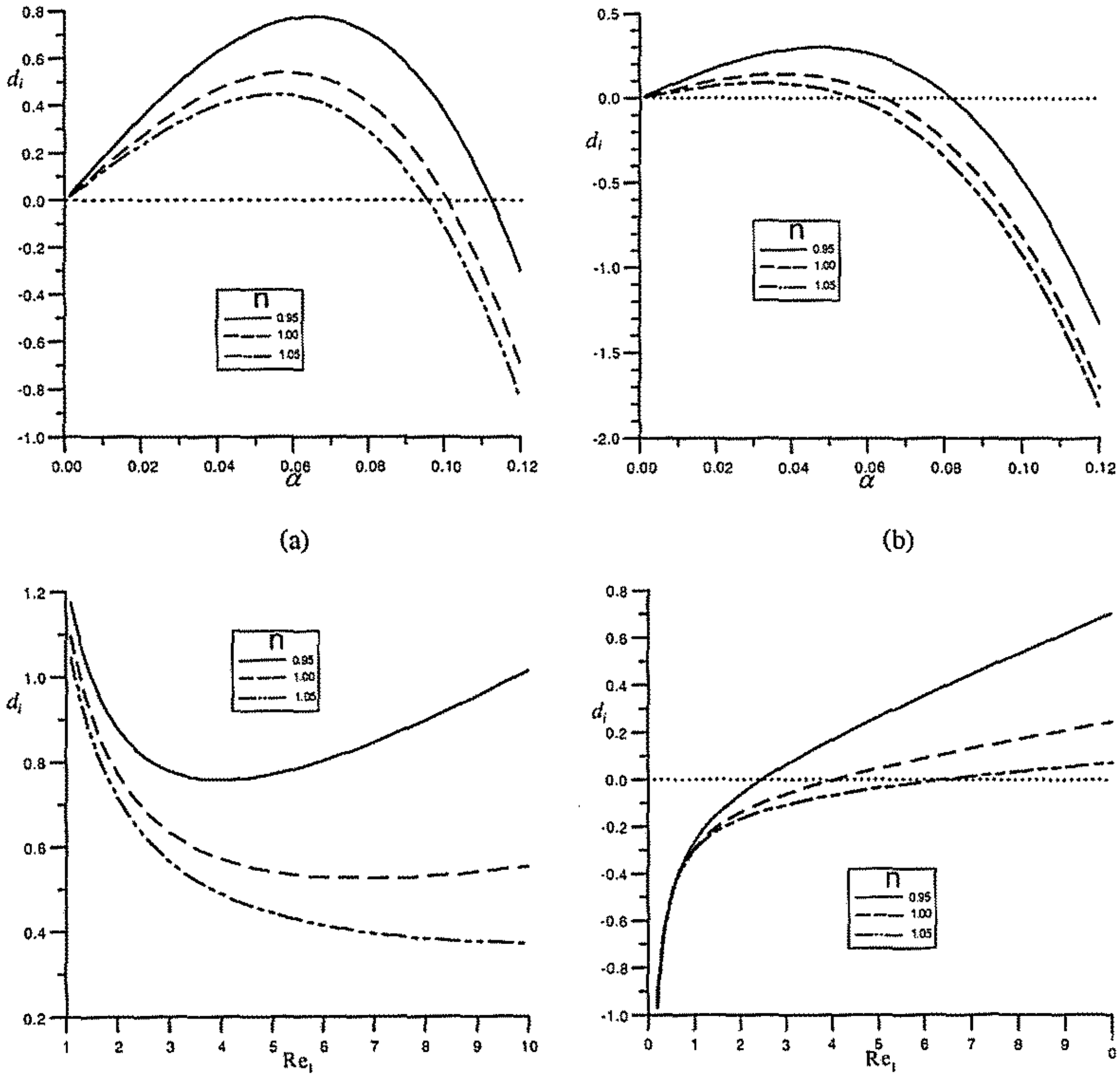


Fig. 2 Neutral stability curves for (a) three different n values at R=10. (b) three different n values at R=20. (c) four different R values at n=0.95. (d) four different R values at n=1.05

Figures 3(a)-3(d) show the temporal film growth rate of the power law fluid for n=0.95 and n=1.05. The temporal film growth rate of the Newtonian flow (i.e. n=1.00) is also given in the figure for comparison purpose. It is interesting to note that temporal film growth rate increases as the values of n decreases. Furthermore, it is found that both the wave number of neutral mode and the maximum temporal film growth rate increase as the value of n decreases. In other words, the larger the value of flow index n is, the higher the stability of a liquid film becomes. Figures 4(a)-4(d) show the temporal film growth rate of power law fluid for R=10,20 and 50. The temporal film growth rate of the plane flow (i.e. R $=\infty$) is also given in the figure for comparison purpose. It is noted that temporal film growth rate decreases as the value of R increases. Furthermore, it is found that both the wave number of neutral mode and the maximum temporal film growth rate increase as the value of R decreases. In other words, the larger the value of radius R is, the higher the stability of a liquid film becomes. Otherwise, the degree of instability in the film flow is clearly intensified by the lateral curvature of cylinder for R=10. The film flow is unstable in nearly all computation domain. The wave speed, given by Eq. (51), is a constant value for all wave numbers and Reynolds number. However, the wave speed can be influenced by the flow index n and the radius of the cylinder R. The wave speed in supercritical region for various R and n values is presented in Table 1. It is found that the wave speed decreases with an increasing n and a decreasing R.



(a) (b) (c) (d)

Fig. 3 Amplitude growth rate of disturbed waves in power law flows for three different n values at (a) Re_1=5 and R=10. (b) Re_1=5 and R=20. (c) $\alpha = 0.06$ and R=10. (d) $\alpha = 0.06$ and R=20.

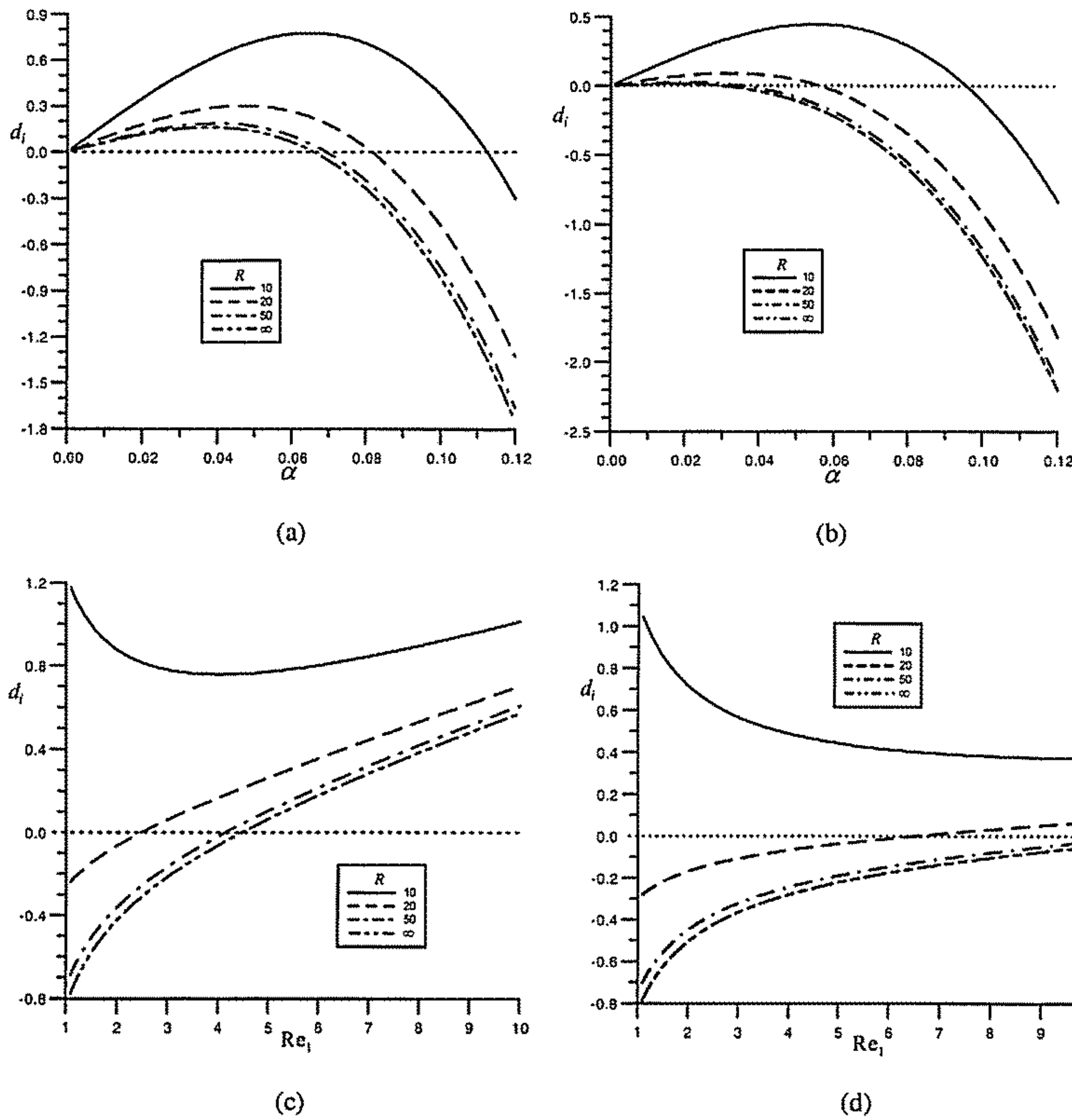


Fig. 4 Amplitude growth rate of disturbed waves in power law flows for four different R values at (a) $\mathrm{Re}_l=5$ and n=0.95. (b) $\mathrm{Re}_l=5$ and n=1.05. (c) $\alpha = 0.06$ and n=0.95. (d) $\alpha = 0.06$ and n=1.05.

It is also noted that a cylinder with a smaller radius makes the flow relatively more unstable. This is due to the surface tension of the lateral curvature. In Eq. (22), the streamwise surface tension term, $S_n \cdot \mathrm{Re}^{\frac{-3n^2+4n+4}{(n+2)(n-2)}}(2\Gamma)^{\frac{(3n-2)n}{n+2}}\alpha^2 h_{zz}$, is independent of the value of r. However, the lateral surface tension term, $S_n \cdot \mathrm{Re}^{\frac{-3n^2+4n+4}{(n+2)(n-2)}}(2\Gamma)^{\frac{(3n-2)n}{n+2}} r^{-1}$, is inverse to the value of r. When the film flows down the outer surface of the cylinder with a smaller radius, the surface tension term of the lateral curvature will become larger. Therefore, it has a destabilizing effect. This destabilizing effect occurs because the radius of the trough of waves have a smaller value than that at the crest of the waves, and the surface tension will produce large capillary pressure at a smaller radius of curvature. This will induce the capillary pressure and force the fluid in the through to move upward to the crest. Thus, the amplitude of the wave is increased.

As discussed above, the degree of stability in a power law film flow is positively proportional to the values of both n and R except that the onset of azimuthal instability occurs beyond finite flow rates and the range of waves stable to azimuthal disturbances become larger. Though the curvature reduces the stability of the flat-film basic state against axisymmetric disturbances, its effect for enhancing the stability of finite-amplitude axisymmetric waves against azimuthal disturbances is found also when small radii are considered [17]. By setting $R \to \infty$, the result becomes a solution for the plane flow problem. In the plane flow solution, it is noted that the flow field becomes relatively stable as the flow index n increases. This phenomenon agrees well with the conclusion given by several authors [21, 23, 24]. By setting n=1.00, the results of a classical Newtonian flow are obtained. As compared to the analysis results given by Hung et al. [15], it is found that both solutions agree well with each other.

Table 1. Wave speed in supercritical region for Various R and n values

n \ d_r \ R	10	20	50	∞
0.95	2.07	2.10	2.12	2.14
1.00	1.94	1.97	1.99	2.00
1.05	1.83	1.85	1.87	1.88

5. CONCLUDING REMARKS

The stability of a power law thin film flow traveling down along a vertical cylinder is thoroughly investigated in this paper by using the method of long wave perturbation. The generalized nonlinear kinematic equation of the free film surface near the wall is derived and is numerically estimated to study the stability of flow field under different values of flow index and radius of the cylinder. Based on the analysis results, several conclusions can be made as follows:

1. In the stability analysis, the neutral stability curve that separates the flow field into two different regions was computed for the amplitude growth rate of $d_i = 0$.
2. The analysis results indicate that the area of unstable region becomes larger for a decreasing n and a decreasing R.

3. It is also noted that the temporal film growth rate is reduced with an increasing n and an increasing R.
4. The wave speed in the supercritical region decrease with an increasing n value and a decreasing R value.
5. The values of flow index n and radius R strongly affect the stability characteristic of a flow film. It is generally true that the stability of a power law film flow increases as the value of n increases and the value of R increases. The azimuthal disturbance appears much more significantly with larger radius. However, in this study, the radii are confined under a selected value.

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NOMENCLATURE

d	complex wave celerity $= d_r + id_i$
g	gravitational acceleration
h	film thickness

h_0^*	local base flow film thickness
n	flow index of the power law fluid
p	fluid pressure
p_a^*	pressure of the atmosphere
R	radius of cylinder
Re_n	Reynolds number $= u_0^{*2-n} h_0^{*n} / \nu$
r, z	coordinates transverse and along to the cylinder surface
S_n	surface tension of the fluid
t	time
u_0^*	reference velocity $= (g / \nu_0)^{1/n} h_0^{*1+1/n} /(4\Gamma)$
u, w	velocities along r- and z-directions, respectively

Greek symbols

α	dimensionless wave number
ε	infinitesimal parameter
η	dimensionless perturbed film thickness
λ	perturbed wave length
μ	fluid dynamic viscosity
ν	fluid kinematic viscosity
ρ	density of the fluid
φ	stream function of the fluid

Superscripts

$*$	dimensional quantities
$'$	differentiation with respect to h

Subscripts

t, r, z	partial differentiation with respect to the subscript
0,1,2....	expansion order of the long wave