SYNTHESIS OF PLANAR FOUR-BAR MECHANISMS

Katrin Brunnthaler, Martin Pfurner and Manfred Husty
Institute for Basic Sciences in Engineering
Unit Geometry and CAD
University of Innsbruck, Austria
E-mail: katrin.brunnthaler@uibk.ac.at, martin.pfurner@uibk.ac.at, manfred.husty@uibk.ac.at

Received December 2005, Accepted March 2006
No. 05-CSME-67, E.I.C. Accession 2919

ABSTRACT

Designing a four-bar mechanism that guides a coupler system through five given poses is an old and well known problem named after L. Burmester. In this paper we show that with kinematic mapping a much neater, more comprehensive solution is obtained. It produces a univariate quartic that can be solved explicitly. Furthermore, solutions that yield ordinary four-bars, slider-cranks or elliptical trammels are identified, a-priori.

SYNTHESE DE MECANISMES PLAN A QUATRE BARRES.

RÉSUMÉ

La conception d'un mécanisme à quatre barres dirigeant un système de couplage à travers les cinq poses données soulève inévitablement le problème de Burmester. Dans notre contribution, nous démontrons que l'application cinématique permet d'obtenir une définition à la fois plus précise et plus complète. Elle génère un quartique à une variable qui peut être résolu de façon explicite. A priori, elle permettrait également l'identification de résolutions qui donnent à voir des mécanismes à quatre barres ordinaires, des mécanismes à manivelle et tiroir ou des doubles tiroirs.

1 INTRODUCTION

A planar four-bar mechanism is a closed kinematic chain, which consists of four bars, linked by four revolute joints. One link, called the base, is located in the fixed system Σ_0. It is connected with two links to the coupler, the moving system Σ. Given five finitely separated

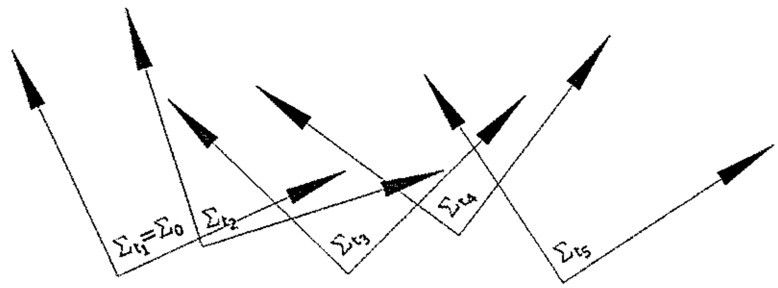

Figure 1: Five given poses

poses (position and orientation) $\Sigma_{t_1}, \dots \Sigma_{t_5}$ of Σ (Fig.1) one can always find a finite set of planar four-bar mechanisms, guiding the coordinate system attached to the coupler through them. Note that not all poses necessarily have to lie in the same assembly branch of the four-bar. An algorithm to detect a branch defect only from the given poses (i.e. before synthesizing the mechanism) was presented recently in [15]. The problem of finding the design parameters of the four-bar when the five poses are given is called the five position Burmester problem, see Burmester [3]. This synthesis problem can be solved exactly due to the fact, that the five poses provide a number of equations equal to the number of variables. There exist a number of different ways to solve this problems, most of them use kinematic properties of the motion itself.

Bottema and Roth [2], McCarthy [13], Lichtenheld [12] and Hunt [9] solved the problem by intersecting the two center point curves to obtain the centers of the revolute joints in the fixed system. The centers of the revolute joints in the moving system are found by intersecting the two circle point curves. These points represent the points moving on circles in the synthesized four-bar motion. Bottema and Roth [2] also report on a solution of this problem in a 6-dimensional projective design parameter space by solving a system of two quadratic and four linear equations.

In this paper the problem is solved in closed form using the 3-dimensional projective kinematic image space of planar Euclidean displacements. Kinematic mapping was introduced independently by Blaschke [1] and Grünwald [5] in 1911. The first attempt to solve the Burmester problem using this method is published by Hayes and Zsombor-Murray [7]. Further developments are reported in Hayes et al.[8] and in the master theses of G. Qiao [14] and J. Nie [11]. Unfortunately in these papers only numerical solutions of the derived set of algebraic equations is given. Furthermore the authors do not discuss the design of special four-bars like slider cranks and double sliders. To the best of the authors' knowledge the paper at hand presents for the first time a complete and closed form solution of the Burmester problem using kinematic mapping, including a discussion of all special cases.

In kinematic mapping every displacement of the Euclidean plane is mapped to a point of a 3-dimensional kinematic image space. A one parameter motion is mapped to a curve, a two parameter motion is represented by a surface. As shown in Bottema and Roth [2], the constraint of a point bound to move on a circle with fixed center and radius maps to a hyperboloid of one sheet in the kinematic image space. If the circle degenerates to a line, the hyperboloid degenerates to a hyperbolic paraboloid as shown in Hayes and Husty [6]. In Bottema and Roth this hyperbolic paraboloid is called a special hyperboloid. Therefore the motion of the coupler, constrained by two points moving on circles, degenerate or not, is represented by the intersection of two hyperboloids of one sheet, of a hyperboloid of one sheet and a hyperbolic paraboloid or of two hyperbolic paraboloids in the kinematic image space. For the algebraic setup of the problem we follow Husty [10], who used the approach of Bottema and Roth [2].

The paper is organized as follows: In Section 2 we give a brief introduction to kinematic mapping of planar displacements. Section 3 deals with mechanism analysis. Section 4 establishes the constraint equations for the synthesis problem and presents conditions to distinguish, if among the synthesized mechanisms there exist special four-bars that pass through the five given poses. Furthermore in this section the solution of the synthesis problem in closed form is obtained. Section 5 gives an overview of the synthesis algorithm and Section 6 illustrates the algorithm with numerical examples.

2 PLANAR KINEMATIC MAPPING

Every Euclidean displacement d of a plane Σ can be written as $\mathbf{p_0} = \mathbf{A} \cdot \mathbf{p}$, where $\mathbf{p}$ is a vector whose entries are the homogeneous coordinates of the moving point, expressed in Σ, $\mathbf{p_0}$ represents the same point in the fixed system Σ_0. The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ a & \cos\phi & -\sin\phi \\ b & \sin\phi & \cos\phi \end{pmatrix}, \tag{1}$$

where (a, b) are the components of the translation vector connecting the origins of Σ and Σ_0 and ϕ is the rotation angle of Σ relative to Σ_0. In 1911 W. Blaschke and J. Grünwald simultaneously introduced kinematic mapping as a mapping κ of the planar Euclidean

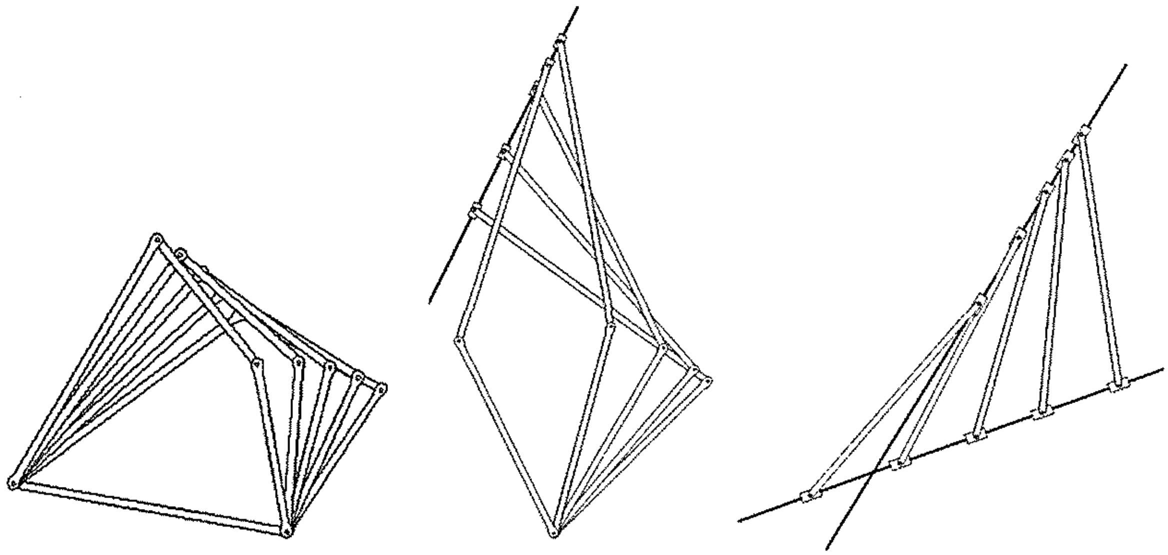

Figure 2: All possible four-bar mechanisms: a general one, a slider crank and a double slider mechanism

displacements ($d \in SE_2$) into a projective 3-space by

$$\kappa : SE_2 \mapsto P^3$$
$$d \mapsto \kappa(d) = \left(2\cos\frac{\phi}{2} : 2\sin\frac{\phi}{2} : a\sin\frac{\phi}{2} - b\cos\frac{\phi}{2} : a\cos\frac{\phi}{2} + b\sin\frac{\phi}{2}\right) = (X_0 : X_1 : X_2 : X_3). \tag{2}$$

Every point in the kinematic image space, except those, where $X_0 = X_1 = 0$, corresponds to a unique displacement of the plane. Given a point in the image space the entries of $\mathbf{A}$ can be computed with help of the following relations:

$$\tan\frac{\phi}{2} = \frac{X_1}{X_0}, \quad a = \frac{2(X_1X_2 + X_0X_3)}{X_0^2 + X_1^2}, \quad b = \frac{2(X_1X_3 - X_0X_2)}{X_0^2 + X_1^2}. \tag{3}$$

Using the expressions in Eq.3 $\mathbf{p}_0 = \mathbf{A} \cdot \mathbf{p}$ can be written with $\mathbf{A}$ in terms of the homogeneous image space coordinates:

$$\begin{pmatrix} Z \\ X \\ Y \end{pmatrix} = \begin{pmatrix} X_0^2 + X_1^2 & 0 & 0 \\ 2(X_1X_2 + X_0X_3) & X_0^2 - X_1^2 & -2X_0X_1 \\ 2(X_1X_3 - X_0X_2) & 2X_0X_1 & X_0^2 - X_1^2 \end{pmatrix} \begin{pmatrix} z \\ x \\ y \end{pmatrix}. \tag{4}$$

In the following we will assume, that all points of interest in the moving system (coupler) are finite. Therefore we set $z = 1$ in Eq. 4.

3 MECHANISM ANALYSIS

In this section the constraint equations representing the constraint of one point moving on a circle are derived. A circle in the fixed system Σ_0 is given by the equation:

$$C_0(X^2+Y^2)-2C_1XZ-2C_2YZ+(C_1^2+C_2^2-R^2)Z^2=0 \tag{5}$$

where $(Z:X:Y)$ are the coordinates of the moving point on the circle, expressed in Σ_0; (C_1,C_2) are the coordinates of the center and R is the radius of the circle. The coefficient C_0 acts as a switch: if the circle degenerates to a line, $C_0=0$, else $C_0=1$. Substituting the values of Z,X,Y from Eq.4 into Eq.5 yields the equation of a surface representing the constraint in the kinematic image space (Bottema and Roth [2]):

$$\begin{aligned}(R^2-C_1^2-C_2^2-C_0(x^2+y^2)+2C_1x+2C_2y)X_0^2\\ +(R^2-C_1^2-C_2^2-C_0(x^2+y^2)-2C_1x-2C_2y)X_1^2\\ +[(4C_2x-4C_1y)X_1+(4C_0y-4C_2)X_2+(-4C_0x+4C_1)X_3]X_0\\ +[(4C_1+4C_0x)X_2+(4C_0y+4C_2)X_3]X_1-4C_0X_3^2-4C_0X_2^2=0.\end{aligned} \tag{6}$$

Given the constants C_1,C_2,R,x,y and $C_0=1$, this quadric surface is a hyperboloid of one sheet, if $C_0=0$ the zero set of Eq.6 represents a hyperbolic paraboloid in the kinematic image space.

4 MECHANISM SYNTHESIS

In the Burmester synthesis problem five poses of a moving system Σ are given. Without loss of generality we can assume that the fixed system Σ_0 coincides with one of these poses[1]. Thus, the image space point, which represents the identity

$$(X_0:X_1:X_2:X_3)=(1:0:0:0) \tag{7}$$

has to be on the constraint quadric Eq.6. Substituting condition (7) into Eq.6 yields an equation for the radius R:

$$R^2-C_1^2-C_2^2-C_0(x^2+y^2)+2C_1x+2C_2y=0. \tag{8}$$

Solving Eq.8 for R and substituting into Eq.6 we obtain the simplified circle constrained equation which will be crucial for the following discussion:

$$\begin{aligned}(-X_0X_3x+X_0X_2y+X_1X_2x+X_3X_1y-X_2^2-X_3^2)C_0-X_0X_2C_2+X_0X_3C_1\\ +X_0X_1xC_2-X_1^2xC_1+X_1X_2C_1-X_0X_1yC_1-X_1^2yC_2+X_1X_3C_2=0\end{aligned} \tag{9}$$

For further simplification we can use the fact that C_0 acts like a switch as mentioned in Section 3. Therefore we can make the following distinction of cases:

[1]Otherwise the application of a unique Euclidean transformation, which does not change the design of the mechanism, will produce this situation.

4.1 Case A: At Least One Point Moves on a Line

Due to the fact that in this case $C_0 = 0$ holds, the constraint equation Eq.9 simplifies to

$$-X_0X_2C_2+X_0X_3C_1+X_0X_1xC_2-X_1^2xC_1+X_1X_2C_1-X_0X_1yC_1-X_1^2yC_2+X_1X_3C_2 = 0. \quad (10)$$

Substituting the image space coordinates of the remaining four poses into this equation yields a system of four bilinear equations in the unknowns C_1, C_2, x, y : [2]

$$[(-C_2x + C_1y)X_{1i} - C_1X_{3i} + C_2X_{2i}]X_{0i} + (C_2y + C_1x)X_{1i}^2 + (-C_2X_{3i} - C_1X_{2i})X_{1i} = 0 \qquad i = 1, \ldots, 4. \quad (11)$$

Geometrically interpreted, these equations represent four quadrics in the design parameter space. If the five given poses can be reached by a slider crank or a double slider mechanism, this system has to have at least one non-trivial solution for the unknowns C_1, C_2, x and y. Note, that (C_1, C_2) is the normal vector of the line l on which the point with the coordinates (x, y) is constrained to move. Therefore only the ratio of C_1 and C_2 is relevant. This implicates that the system is overconstrained.

4.1.1 Subcase A_1: l Is Parallel to the X-Axis of Σ_0

In this case we have $C_1 = 0$ and without loss of generality we can assume $C_2 = 1$. The constraint equations further simplify to:

$$-X_{1i}X_{0i}x + X_{1i}^2y + X_{2i}X_{0i} - X_{3i}X_{1i} = 0, \qquad i = 1, \ldots, 4. \quad (12)$$

Geometrically these equations represent four lines in Σ. The system (12) is overconstrained and an easy consideration shows that it has solutions iff the four lines intersect in a point. Therefore a planar four-bar mechanism with at least one P-joint, having the axis parallel to the X-axis of Σ_0 and guiding a rigid body through the given poses can only exist, when the four lines in the set of equations (12) are in a pencil of lines. Solving any two equations of this system for x, y and back substituting the solutions into the other two equations yields two compatibility conditions E_1 and E_2:

$$E_1 : \quad \left(-\frac{X_{13}(-X_{11}^2X_{02}X_{22}+X_{01}X_{21}X_{12}^2-X_{11}X_{31}X_{12}^2+X_{11}^2X_{12}X_{32})}{X_{11}X_{12}(X_{01}X_{12}-X_{11}X_{02})} + X_{23}\right)X_{03}$$
$$-\frac{(X_{11}X_{31}X_{02}X_{12}-X_{01}X_{11}X_{12}X_{32}-X_{01}X_{21}X_{02}X_{12}+X_{01}X_{11}X_{02}X_{22})X_{13}^2}{X_{12}X_{11}(X_{01}X_{12}-X_{11}X_{02})} - X_{13}X_{33} = 0 \ , \quad (13)$$

$$E_2 : \quad \left(-\frac{X_{14}(-X_{11}^2X_{02}X_{22}+X_{01}X_{21}X_{12}^2-X_{11}X_{31}X_{12}^2+X_{11}^2X_{12}X_{32})}{X_{11}X_{12}(X_{01}X_{12}-X_{11}X_{02})} + X_{24}\right)X_{04}$$
$$-\frac{(X_{11}X_{31}X_{02}X_{12}-X_{01}X_{11}X_{12}X_{32}-X_{01}X_{21}X_{02}X_{12}+X_{01}X_{11}X_{02}X_{22})X_{14}^2}{X_{12}X_{11}(X_{01}X_{12}-X_{11}X_{02})} - X_{34}X_{14} = 0 \ . \quad (14)$$

[2]In Eq.11 the image space coordinates of the ith pose were denoted by $(X_{1i} : X_{2i} : X_{3i} : X_{4i})$.

4.1.2 Subcase A_2: l Is Not Parallel to the X-Axis of Σ_0

In this case we can set $C_1 = 1$. This yields four constraint equations of the form:

$$[(-C_2x + y)X_{1i} - X_{3i} + C_2X_{2i}]X_{0i} + (C_2y + x)X_{1i}^2 + (-C_2X_{3i} - X_{2i})X_{1i} = 0 \qquad i = 1, \ldots, 4. \tag{15}$$

Solving any two equations of the set linearly for x and y and substituting the solutions into one of the remaining equations one ends up with an equation of degree three in C_2, which can be factored. Two solutions of this equation are always complex ($C_2 = \pm I$), and do not correspond to real mechanisms. The other factor of the equation can be solved linearly for C_2. Substituting the solutions for x, y and C_2 into the remaining fourth equation yields a compatibility condition E_3, which is displayed in the Appendix.

Remark 1. Geometrically subcases A_1 and A_2 are of course identical. They only have to be distinguished because of the choice of the fixed coordinate system. (see figures in Sections 6.2.1 and 6.2.2)

4.2 Case B: At Least One Point Moves on a Circle

This is the general case and due to the fact, that in this case we can set $C_0 = 1$, the constraint equation simplifies to

$$\begin{aligned}((-C_1y + C_2x)X_1 + (y - C_2)X_2 + (C_1 - x)X_3)X_0 + (-C_2y - C_1x)X_1^2 \\ +((x + C_1)X_2 + (y + C_2)X_3)X_1 - X_3^2 - X_2^2 = 0.\end{aligned} \tag{16}$$

Following Gfrerer [4] we apply a coordinate transformation for further simplification of the equations and set:

$$\begin{aligned} -x - C_1 &= 2b_1 \qquad & y - C_2 &= 2b_3 \\ -y - C_2 &= 2b_2 \qquad & -x + C_1 &= 2b_4. \end{aligned} \tag{17}$$

Applying this linear coordinate transformation, the circle constraint equation rewrites to

$$\begin{aligned}[(2b_1b_3 + 2b_2b_4)X_1 + 2b_3X_2 + 2b_4X_3]X_0 + (-b_1^2 - b_2^2 + b_3^2 + b_4^2)X_1^2 \\ +(-2b_1X_2 - 2b_2X_3)X_1 - X_2^2 - X_3^2 = 0.\end{aligned} \tag{18}$$

Eq.18 is the most simplified version of the circle constrained equation and will be used to derive a closed form solution of the general synthesis problem. Substituting the image space coordinates $(X_{1i} : X_{2i} : X_{3i} : X_{4i})$ of the ith pose into this equation yields the following system of constraint equations:

$$\begin{aligned}[(2b_1b_3 + 2b_2b_4)X_{1i} + 2b_3X_{2i} + 2b_4X_{3i}]X_{0i} + (-b_1^2 - b_2^2 + b_3^2 + b_4^2)X_{1i}^2 \\ +(-2b_1X_{2i} - 2b_2X_{3i})X_{1i} - X_{2i}^2 - X_{3i}^2 = 0 \\ i = 1, \ldots, 4.\end{aligned} \tag{19}$$

The quadratic terms of the unknowns in the constraint equations (19) can be eliminated by simple manipulations (multiplication of equations with constant factors and subtractions of different ones). This yields three bilinear equations in the unknowns b_1, b_2, b_3, b_4. Performing the same process once more, one can get rid of the bilinear terms and two equations linear in the unknowns remain. Solving these equations for two of the unknowns, e.g. b_1, b_2, gives these two as functions of the other unknowns. Taking one of the bilinear equations and one out of the system (19) (now dependent just on b_3, b_4) and calculating the resultant yields an equation of degree four in e.g. b_4, which is too long to be displayed here. This equation can be solved in closed form (see L. Ferrari (1522-1565)). It has 0, 2 or 4 real roots for b_4. Substituting one of this values in the two equations used for the resultant yields two quadratic equations in b_3. Because of the fact, that these two equations are now redundant, we can eliminate b_3^2 to produce one linear equation in b_3 and solve for this unknown. We can use the same procedure to obtain one value of b_3 for each real root of b_4. As b_1 and b_2 are linear functions of b_3 and b_4 they can be calculated easily.

With the inverse coordinate transformation to Eq.17 we obtain for each solution $b_1, \ldots, b_4$ the coordinates of the center (C_1, C_2), which is one of the Burmester points, and the coordinates of the point (x, y) constrained to move on the circle.

5 SYNTHESIS ALGORITHM

Initially, five poses of the moving system Σ are given as points in the kinematic image space with homogeneous coordinates $(\bar{X}_{0i} : \bar{X}_{1i} : \bar{X}_{2i} : \bar{X}_{3i})$ for $i = 1, \ldots, 5$. To obtain the situation, that one of the given points is the origin, one has to apply a fixed coordinate transformation to all of the given points. Note that such a fixed coordinate transformation does not change the kinematic situation. In the following we assume that this transformation has been performed and we refer just to the remaining four given points in the kinematic image space $(X_{0i} : X_{1i} : X_{2i} : X_{3i})$ for $i = 1, \ldots, 4$. But one always has to keep in mind, that the identity is then the remaining fifth Burmester position.

To determine if all positions of a resulting mechanism lie in a common assembly mode, one applies the branching defect detection algorithm of Schröcker, Husty and McCarthy [15].

Afterwards we have to test if among the Burmester points we are searching for there are points at infinity. This means, that the corresponding point (x, y) of the coupler moves on a degenerated circle, a line and the synthesized RR chain is a slider. Therefore we have to substitute the values of the four given image space points into the equations E_1 (Eq.13), E_2 (Eq.14) and E_3 (Eq.9 in the Appendix). The different possible cases are dealt with in the subsections below.

In a next step it is necessary to plug the input data into the general case Equation (19) to obtain the finite solutions.

5.1 E_1, E_2 and E_3 Are Satisfied

If all three conditions are satisfied we know that at least two Burmester points are at infinity. One of them is the point at infinity of the Y-Axis of the fixed frame Σ_0. Such a motion, realized by a double slider, is a so called cardan motion (elliptical trammel). It also can

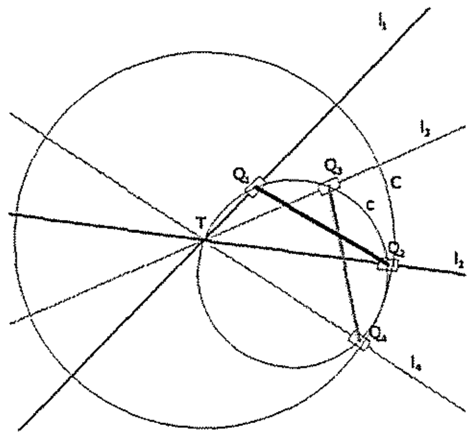

Figure 3: Double slider and replacement of lines

be modelled by rolling a circle c in another circle C having double radius. The center of C is then the point of intersection T of the two given lines l_1, l_2, representing the axes of the sliders. c is on T and on the two moving points Q_1, Q_2. It is well known that the path of every point of c in the rolling motion is a line passing through T. One can replace the two given lines (l_1, l_2) with any two other lines (l_3, l_4) having the same point of intersection T, as long as the moving points (Q_1, Q_2) and (Q_3, Q_4) lie on the same circle c (Fig.3). This verifies, that one can always find a pair of axes for the P-joints with one line parallel to the X-axes of Σ_0. It is easy to see that the center of c moves on a circle with the same radius centered in T. Therefore, as shown in Wunderlich [16], a cardan motion can also be realized by a slider crank, combining one of the possible sliders with this mentioned circle. Note that C and c are the fixed and moving axodes of both motions. In this case one obtains infinitely many double sliders and slider cranks, realizing always the same motion.

5.2 Only E_1 and E_2 Are Satisfied

If this is the case only one Burmester point is a point at infinity. The axis of the corresponding slider is parallel to the X-axes of Σ_0. Due to the fact that this is only one out of four Burmester points we additionally have to do the computations described in Subsection (4.2), which yield four circles, but one of them has infinite radius and corresponds to the axis of the slider. Assuming that all solutions are real we obtain in this case three general four-bars and three slider cranks passing through the five given poses.

5.3 Only E_3 Is Satisfied

In this case the synthesis results in one slider with an axis not parallel to the X-axes of Σ_0. The other Burmester points can be found in the same way as in the subsection before. The geometric interpretation is analogous to the subsection before: assuming that all solutions are real we obtain in this case three general four-bars and three slider cranks passing through the five given poses.

5.4 None of the Conditions E_1, E_2, E_3 Is Satisfied

Neither E_1, E_2 nor E_3 is satisfied. This means that the synthesis will yield only finite Burmester points. In this case we obtain 4 RR-dyads, which can be combined to 6 mechanisms.

Note that in all cases listed above either $0, 1$ or 6 mechanisms can be real ($0, 2$ or 4 roots of Eq.19 are real). This concludes the discussion of all possible different cases. In the following a flowchart of the synthesis algorithm is given.

Algorithm 1.

1. Initially five poses of a moving system Σ are given.
2. Apply a coordinate transformation to all given poses such that one of the poses coincides with the fixed coordinate frame.
3. Now only four arbitrary poses of a moving system remain, given by their coordinates $(X_{0i} : X_{1i} : X_{2i} : X_{3i})$ for $i = 1, \ldots, 4$ in the kinematic image space (see Section 2).
4. To determine if all poses lie in the same assembly mode see Schröcker, Husty and McCarthy [15].
5. Substitute the coordinates of the poses in the Equations E1 (Eq.13), E2 (Eq.14) and E3 (see Appendix) to determine if some of the four Burmester points we are searching for are at infinity.
   - E_1, E_2 and E_3 are satisfied: at least two Bourmester points are at infinity and one of them is the point at infinity of the Y-axis of the fixed frame.
   - Only E_1 and E_2 are satisfied: only one Burmester point is a point at infinity. This point is the point at infinity of the X-axes of the fixed frame.
   - Only E_3 is satisfied: only one Burmester point is a point at infinity. This point is not the point at infinity of the X-axes of the fixed frame.
   - None of the conditions E_1, E_2 and E_3 is satisfied: none of the Burmester points is at infinity.
6. Substitute the input data into the general case equation Eq.(19) to obtain the finite solutions.

6 NUMERICAL EXAMPLES

To verify the established theory we have designed several mechanisms and used them to obtain five input poses.[3] The design parameters are listed in Tables 1, 4, 7 and 10. Then the given poses were taken as input to the algorithm described above to synthesize four-bars. Note that in any case the algorithm has to return the input mechanisms and other RR-chains or degenerate RR-chains. The results are presented in tables, which show the synthesized mechanisms, the five chosen poses, the two points moving on circles or lines. In the figures we show several resulting mechanisms with the paths of three points of the moving system during the coupler motion. We mention that the depicted coordinate frames have to be thought rigidly attached to the coupler. For sake of clearness of the figures this connection has been omitted.

6.1 General Four-Bar Mechanism

C_0	1	1
C_1	2	6
C_2	2	1
x	7.3821	9.1605
y	4.2434	1.1070

Table 1: Design parameters of mechanism 1

	pose 1	pose 2	pose 3	pose 4
a	-0.245005	-0.914683	-2.056744	-3.054058
b	0.523260	1.240571	2.235073	3.179009
ϕ	0.101061	0.116316	0.072202	-0.013746

Table 2: Given relative poses

	solution 1	solution 2	solution 3	solution 4
C_0	1	1	1	1
C_1	-34.640483	1.999996	6.000008	-4.402381
C_2	-29.947423	2.000000	0.999996	16.136008
x	18.091483	7.382096	9.160473	-3.697626
y	17.844191	4.243444	1.106973	13.877304
$\Rightarrow R$	71.166696	5.830956	3.162275	2.366097

Table 3: Obtained results

The solutions 2 and 3 are the parameters of the input mechanism. Note that the given poses yield four real solutions. Therefore one can synthesize out of the four RR-dyads six real four-bar mechanisms that guide the coupler system through the given five poses.

[3]Note that in all examples one pose is the identity and therefore will not be listed in the tables.

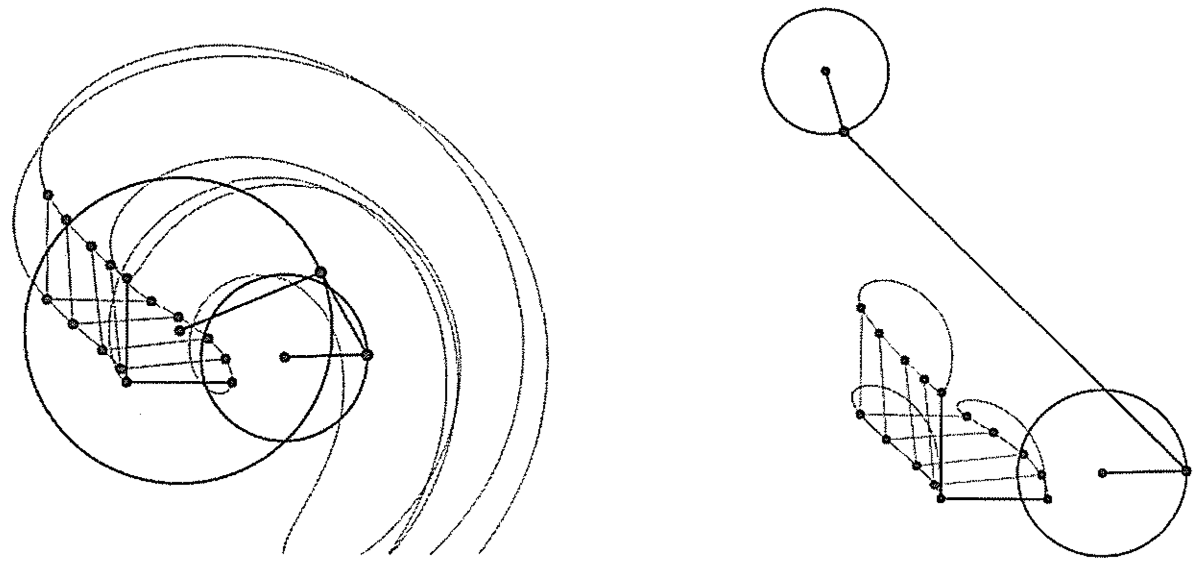

Figure 4: Two out of the six real general four-bar mechanisms that guide Σ through the given poses

6.2 Slider Cranks

6.2.1 l of the Input Mechanism Is Parallel to the X-Axis of Σ_0

C_0	0	1
C_1	0	10
C_2	1	0
x	0	10
y	10	4

Table 4: Design parameters of mechanism 2

	pose 1	pose 2	pose 3	pose 4
a	1.250980	2.246787	3.002000	3.411072
b	0.002722	0.048305	0.386680	1.631868
ϕ	-0.023335	-0.098330	-0.278997	-0.579360

Table 5: Given relative poses

	solution 1	solution 2	solution 3	solution 4
C_0	1	0	1	1
C_1	0.695364	0	8.398954	10.000008
C_2	-57.115519	1	0.158311	-0.000004
x	0.850293	0	8.425612	10.000007
y	11.310480	10	5.257908	3.999990
$\Rightarrow R$	68.426175	∞	5.099667	3.999994

Table 6: Obtained results

As one can see in Table 6: solutions 2 and 4 yield the given input slider crank mechanism. There are again four real solutions, only one solution corresponds to a slider. The other solutions are cranks. Therefore one can design three slider cranks and three general four-bars that guide the coupler through the given five poses.

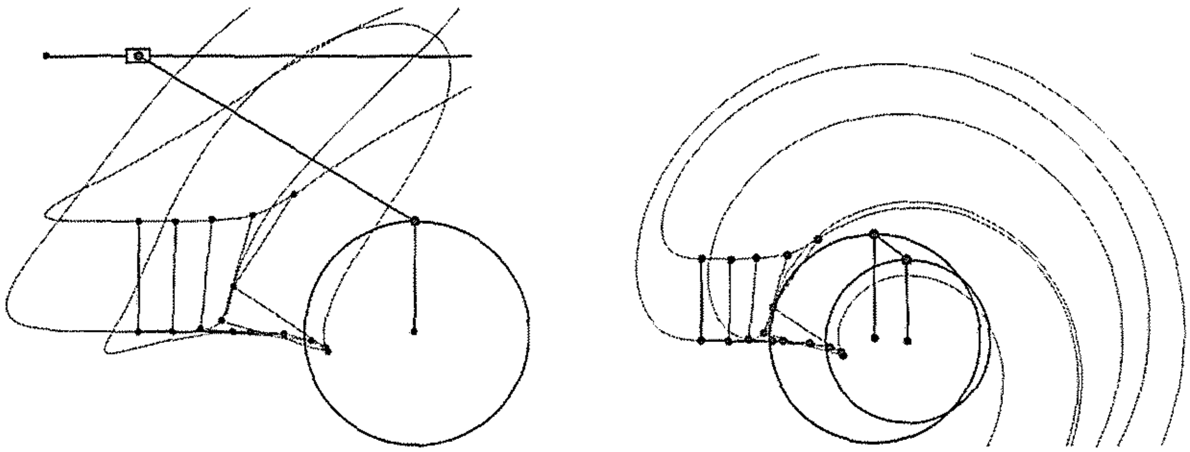

Figure 5: A slider crank and a general four-bar mechanism that guide Σ through the given poses

6.2.2 l of the Starting Mechanism Is Not Parallel to the X-Axis of Σ_0

C_0	0	1
C_1	1	1
C_2	-1	3
x	10	1
y	3	8

Table 7: Design parameters of mechanism 3

	pose 1	pose 2	pose 3	pose 4
a	2.876039	6.140158	7.937402	9.480527
b	-0.146268	-0.300979	-0.469344	-0.719109
ϕ	0.221168	0.456081	0.589363	0.712381

Table 8: Given relative poses

	solution 1	solution 2	solution 3	solution 4
C_0	0	1	1	1
C_1	1	1.000952	1.153186	53.908851
C_2	-1	3.000680	3.112102	-17.427364
x	9.99999	1.000438	1.071432	9.746250
y	2.99999	7.999951	7.993329	7.184279
$\Rightarrow R$	∞	4.999271	4.881911	50.557574

Table 9: Obtained results

Apart of the fact, that the axis of the slider is not parallel to the X-axis of Σ_0, the results of this case are analogous to the results of the previous case.

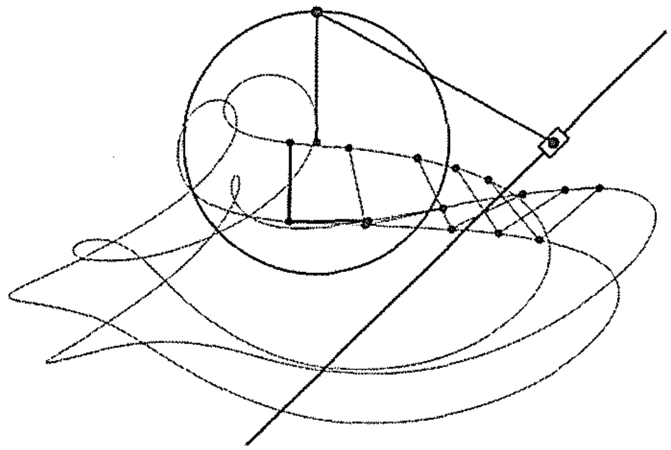

Figure 6: One of the three slider cranks

6.3 Double Slider Mechanism

In this case we obtain three solutions (Tab.12): solutions 1 and 2 correspond to the input sliders. Solution number three is a crank. As expected, the center of the third solution is the point of intersection of the lines of the input mechanism, the moving point is the center of c.

C_0	0	0
C_1	1	0
C_2	0	1
x	3	8
y	8	3

Table 10: Design parameters of mechanism 4

	pose 1	pose 2	pose 3	pose 4
a	2.855481	5.699623	8.102182	9.891484
b	-2.508287	-4.364129	-5.350390	-5.513084
ϕ	0.341855	0.680246	0.998733	1.297126

Table 11: Given relative poses

	solution 1	solution 2	solution 3
C_0	0	0	1
C_1	0	1	2.999984
C_2	1	-1.67568	3.000000
x	8	8.88721	5.499997
y	3	6.51333	5.500004
$\Rightarrow R$	∞	∞	3.535545

Table 12: Obtained results

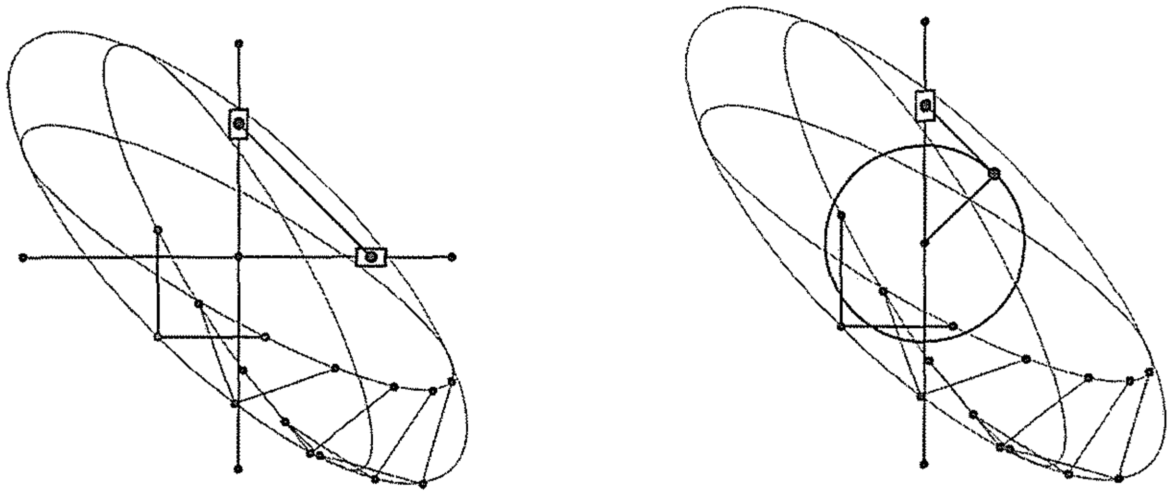

Figure 7: A double slider and a slider crank mechanism that guide Σ through the given poses

7 CONCLUSION

In this paper a complete and closed form solution of the five position Burmester problem using kinematic mapping was presented. As a new result we have found three equations that will determine if there are sliders among the RR-chains to be synthesized. In detail all possible combinations of sliders, elliptical trammels and general four-bars were classified and discussed. The theoretical results were illustrated by a number of numerical examples. The investigation of the geometrical structure of the design parameter space and the meaning of equations E_1, E_2 and E_3 is subject of further research.

8 ACKNOWLEDGMENT

The authors would like to thank Prof. Paul Zsombor-Murray for fruitful discussions that helped to improve this manuscript.

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9 APPENDIX

$$
\begin{aligned}
E_3: \quad & (X_{14}X_{04}X_{01}X_{02}X_{31}X_{13}^2X_{22} - X_{34}X_{14}X_{03}X_{13}X_{12}^2X_{21}X_{11} - X_{34}X_{04}X_{13}^2X_{11}X_{01}X_{02}X_{22} - X_{34}X_{04}X_{13}X_{11}X_{12}^2X_{33}X_{01} \\
& + X_{34}X_{04}X_{03}X_{23}X_{11}X_{12}^2X_{01} + X_{34}X_{04}X_{13}^2X_{11}X_{01}X_{12}X_{32} + X_{34}X_{04}X_{13}^2X_{02}X_{12}X_{01}X_{21} - X_{34}X_{04}X_{03}X_{13}X_{12}^2X_{01}X_{21} \\
& + X_{34}X_{04}X_{13}X_{11}^2X_{12}X_{33}X_{02} - X_{34}X_{04}X_{03}X_{23}X_{11}^2X_{12}X_{02} + X_{34}X_{04}X_{03}X_{13}X_{11}^2X_{02}X_{22} - X_{34}X_{04}X_{03}X_{13}X_{11}^2X_{12}X_{32} \\
& - X_{34}X_{04}X_{13}^2X_{02}X_{12}X_{11}X_{31} + X_{34}X_{04}X_{03}X_{13}X_{12}^2X_{11}X_{31} - X_{14}^2X_{11}X_{02}X_{32}X_{31}X_{03}X_{13} - X_{14}^2X_{11}X_{12}X_{03}X_{31}X_{13}X_{22} \\
& + X_{14}^2X_{11}X_{12}X_{32}X_{03}X_{21}X_{13} - X_{14}X_{04}X_{01}X_{02}X_{32}X_{21}X_{13}^2 + X_{14}X_{04}X_{01}X_{12}^2X_{13}X_{23}X_{21} + X_{14}X_{04}X_{01}X_{12}^2X_{13}X_{31}X_{33} \\
& - X_{14}X_{04}X_{01}X_{12}^2X_{31}X_{03}X_{23} + X_{14}X_{04}X_{01}X_{12}^2X_{03}X_{33}X_{21} - X_{14}X_{04}X_{01}X_{12}X_{21}X_{13}^2X_{22} - X_{14}X_{04}X_{01}X_{12}X_{31}X_{32}X_{13}^2 \\
& - X_{14}X_{04}X_{11}^2X_{02}X_{22}X_{23}X_{13} - X_{14}X_{04}X_{11}^2X_{02}X_{22}X_{03}X_{33} - X_{14}X_{04}X_{11}^2X_{02}X_{32}X_{33}X_{13} + X_{14}X_{04}X_{11}^2X_{02}X_{32}X_{03}X_{23} \\
& - X_{14}X_{04}X_{11}^2X_{12}X_{22}X_{33}X_{13} + X_{14}X_{04}X_{11}^2X_{12}X_{22}X_{03}X_{23} + X_{14}X_{04}X_{11}^2X_{12}X_{32}X_{23}X_{13} + X_{14}X_{04}X_{11}^2X_{12}X_{32}X_{03}X_{33} \\
& + X_{14}X_{04}X_{11}X_{02}X_{21}X_{13}^2X_{22} + X_{14}X_{04}X_{11}X_{02}X_{31}X_{32}X_{13}^2 + X_{14}X_{04}X_{11}X_{12}^2X_{13}X_{33}X_{21} - X_{14}X_{04}X_{11}X_{12}^2X_{13}X_{31}X_{23} \\
& - X_{14}X_{04}X_{11}X_{12}^2X_{21}X_{23}X_{03} - X_{14}X_{04}X_{11}X_{12}^2X_{31}X_{03}X_{33} + X_{14}X_{04}X_{11}X_{12}X_{31}X_{13}^2X_{22} - X_{14}X_{04}X_{11}X_{12}X_{32}X_{21}X_{13}^2 \\
& - X_{24}X_{14}X_{13}^2X_{11}X_{01}X_{02}X_{22} - X_{24}X_{14}X_{13}X_{11}X_{12}^2X_{33}X_{01} + X_{24}X_{14}X_{03}X_{23}X_{11}X_{12}^2X_{01} + X_{24}X_{14}X_{13}^2X_{11}X_{01}X_{12}X_{32} \\
& + X_{24}X_{14}X_{13}^2X_{02}X_{12}X_{01}X_{21} - X_{24}X_{14}X_{03}X_{13}X_{12}^2X_{01}X_{21} + X_{24}X_{14}X_{13}X_{11}^2X_{12}X_{33}X_{02} - X_{24}X_{14}X_{03}X_{23}X_{11}^2X_{12}X_{02} \\
& + X_{24}X_{14}X_{03}X_{13}X_{11}^2X_{02}X_{22} - X_{24}X_{14}X_{03}X_{13}X_{11}^2X_{12}X_{32} - X_{24}X_{14}X_{13}^2X_{02}X_{12}X_{11}X_{31} + X_{24}X_{14}X_{03}X_{13}X_{12}^2X_{11}X_{31} \\
& - X_{34}X_{14}X_{13}^2X_{11}X_{01}X_{02}X_{32} + X_{34}X_{14}X_{13}X_{11}X_{12}^2X_{23}X_{01} + X_{34}X_{14}X_{03}X_{33}X_{11}X_{12}^2X_{01} - X_{34}X_{14}X_{13}^2X_{11}X_{01}X_{12}X_{22} \\
& + X_{34}X_{14}X_{13}^2X_{02}X_{12}X_{31}X_{01} - X_{34}X_{14}X_{03}X_{13}X_{12}^2X_{31}X_{01} - X_{34}X_{14}X_{13}X_{11}^2X_{12}X_{23}X_{02} - X_{34}X_{14}X_{03}X_{33}X_{11}^2X_{12}X_{02} \\
& + X_{34}X_{14}X_{03}X_{13}X_{11}^2X_{02}X_{32} + X_{34}X_{14}X_{03}X_{13}X_{11}^2X_{12}X_{22} + X_{34}X_{14}X_{13}^2X_{02}X_{12}X_{21}X_{11} + X_{24}X_{04}X_{03}X_{13}X_{12}^2X_{21}X_{11} \\
& + X_{24}X_{04}X_{13}^2X_{11}X_{01}X_{02}X_{32} + X_{24}X_{04}X_{13}^2X_{11}X_{01}X_{12}X_{22} - X_{24}X_{04}X_{13}^2X_{02}X_{12}X_{31}X_{01} - X_{24}X_{04}X_{13}^2X_{02}X_{12}X_{21}X_{11} \\
& + X_{24}X_{04}X_{03}X_{33}X_{11}^2X_{12}X_{02} - X_{24}X_{04}X_{03}X_{13}X_{11}^2X_{02}X_{32} + X_{24}X_{04}X_{03}X_{13}X_{12}^2X_{31}X_{01} - X_{24}X_{04}X_{03}X_{13}X_{11}^2X_{12}X_{22} \\
& - X_{24}X_{04}X_{03}X_{33}X_{11}X_{12}^2X_{01} + X_{34}X_{04}X_{13}X_{11}^2X_{12}X_{23}X_{02} - X_{24}X_{04}X_{13}X_{11}X_{12}^2X_{23}X_{01} + X_{14}^2X_{01}X_{11}X_{02}X_{22}X_{23}X_{13} \\
& + X_{14}^2X_{01}X_{11}X_{02}X_{22}X_{03}X_{33} + X_{14}^2X_{01}X_{11}X_{02}X_{32}X_{33}X_{13} - X_{14}^2X_{01}X_{11}X_{02}X_{32}X_{03}X_{23} + X_{14}^2X_{01}X_{11}X_{12}X_{22}X_{33}X_{13} \\
& - X_{14}^2X_{01}X_{11}X_{12}X_{22}X_{03}X_{23} - X_{14}^2X_{01}X_{11}X_{12}X_{32}X_{23}X_{13} - X_{14}^2X_{01}X_{11}X_{12}X_{32}X_{03}X_{33} - X_{14}^2X_{01}X_{02}X_{12}X_{13}X_{31}X_{33} \\
& - X_{14}^2X_{01}X_{02}X_{12}X_{13}X_{21}X_{23} - X_{14}^2X_{01}X_{02}X_{12}X_{21}X_{03}X_{33} + X_{14}^2X_{01}X_{02}X_{12}X_{31}X_{03}X_{23} - X_{14}^2X_{01}X_{02}X_{03}X_{31}X_{13}X_{22} \\
& + X_{14}^2X_{01}X_{02}X_{32}X_{03}X_{21}X_{13} + X_{14}^2X_{01}X_{12}X_{03}X_{21}X_{13}X_{22} + X_{14}^2X_{01}X_{12}X_{32}X_{03}X_{31}X_{13} - X_{14}^2X_{11}X_{02}X_{12}X_{13}X_{33}X_{21} \\
& + X_{14}^2X_{11}X_{02}X_{12}X_{13}X_{31}X_{23} + X_{14}^2X_{11}X_{02}X_{12}X_{21}X_{23}X_{03} + X_{14}^2X_{11}X_{02}X_{12}X_{31}X_{03}X_{33} - X_{14}^2X_{11}X_{02}X_{21}X_{03}X_{13}X_{22}) = 0
\end{aligned}
$$