FINITE ELEMENT CALCULATIONS FOR INCOMPRESSIBLE MATERIALS USING A MODIFIED LU DECOMPOSITION

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ABSTRACT

An approach for stabilising the finite element analysis of incompressible materials is presented. Although many effective methods exist for addressing incompressibility when applied forces or stress tractions are known, this paper explores cases where displacements are the only prescribed quantities on the boundary. In particular, the standard u-p formulation with isoparametric, quadrilateral, constant pressure, plane strain elements is considered. It is well known that these elements suffer from extreme numerical instability under displacement boundary conditions. A modified LU decomposition is used to filter the equations to produce a reduced nonsingular system leading to determination of the von Mises stress. The method is demonstrated with two test cases commonly used to assess element performance: a rectangular block in a uniform state of plane strain; and an axially restrained thick-walled cylinder subjected to internal and external pressures.

CALCULS PAR ÉLÉMENTS FINIS POUR MATÉRIAUX INCOMPRESSIBLES AVEC UNE DÉCOMPOSITION LU MODIFIÉE

RESUME

Une approche pour stabiliser l'analyse par éléments finis de matériaux incompressibles est présentée. Bien que beaucoup de méthodes efficaces existent pour adresser l'incompressibilité lorsque les forces ou les contraintes appliquées sont connues, cet article explore des cas où les déplacements sont les seules quantités prescrites sur le bord. En particulier, la formulation "u-p" standard pour éléments isoparamétriques, quadrilatéraux, à pression constante, en déformations planes est considérée. Il est bien connu que ces éléments souffrent d'instabilité numérique extrême dans le cas de déplacements imposés. Une décomposition LU modifiée est employée pour filtrer les équations pour produire un système non singulier réduit menant à la détermination des contraintes de von Mises. La méthode est démontrée avec deux tests généralement utilisés pour évaluer la performance des éléments: un bloc rectangulaire dans un état uniforme de déformations planes; et un cylindre à parois épaisses axialement retenu soumis à des pressions internes et externes.

INTRODUCTION

A number of structures can be accurately modelled as consisting of incompressible material. In particular, incompressibility is often one of the underlying assumptions made in performing finite element analyses of soft tissues (See, for example, [1-3]). These analyses are typically performed using mixed formulations [4–6], characterized by a decomposition of the stress field into hydrostatic and deviatoric components, accompanied by kinematic constraints enforcing the constant volume condition. In these "u-p formulations", the hydrostatic stress component, p, is removed from the constitutive law and is determined, like displacement, u, as an independent quantity which must satisfy equilibrium, compatibility, and boundary conditions. Success depends on consistency and numerical stability to guarantee that calculated values converge to a unique accurate solution as element size is decreased. Convergence properties are normally assessed using the Babuška-Brezzi (B-B) criterion [6,7] or the patch test [8] which are essentially equivalent [7]. The patch test includes a count condition for stability assessment where the number of u-degrees of freedom must be greater than or equal to the number of p-degrees of freedom [7]. Both the patch test and the B-B criterion have been used extensively [4-10] and the u-p formulation in its various forms has been applied to numerous problems where a structure is subjected to applied forces or stress tractions [3,11-14].

There are a number of important situations, however, where the applied forces are unknown, and only displacements on the boundary can be measured or specified. One example is the finite element modelling of plastic surgery [15]. In a given procedure, the geometry of incision and suture patterns is known, but the surgeon is unaware of the forces required to achieve these displacements. Another example is the finite element modelling of soft tissues within the body. This can be performed using high resolution imaging, such as with synchrotron light, to identify specific traceable landmarks in the tissue [16]. The data from these images may be supplied to finite element codes for stress analysis and predictive modelling. For each of these examples and other cases with complete Dirichlet boundary conditions, the unknown hydrostatic pressure field is indeterminate and the final reduced system of equations produced by a u-p formulation is singular, or near-singular due to round-off errors, leading to failure of the element according to either the patch test or the Babuška-Brezzi (B-B) criterion. In particular, the near-singular case is problematic since ill-conditioned matrices can lead to catastrophic errors in iterative solutions to nonlinear problems. In such cases, the unknown displacements, and therefore important quantites such as the von Mises stress, theoretically, may be determined [17]. There is a need, however, to develop stable algorithms. The method proposed here provides a step towards lowering the condition number of the coefficient matrix to acceptable levels for intensive calculations.

THE PROPOSED STRATEGY FOR STABILIZING CALCULATIONS

For displacements specified on the entire boundary, the u-p formulation gives rise to systems of equations of the form

$$\mathbf{K^*x^* = b^*} \qquad \text{or} \qquad \left[\begin{array}{c|c} A & B^T \\ \hline B & 0 \end{array}\right]\left[\begin{array}{c} u \\ \hline p \end{array}\right]=\left[\begin{array}{c} b_1 \\ \hline b_2 \end{array}\right], \tag{1}$$

where u is the vector of unknown displacements, p is the vector of unknown elemental pressure degrees of freedom, $Au + B^T p = b_1$ are the equilibrium equations, and $Bu = b_2$ are the constant volume constraints after application of the boundary conditions. The proposed method is based on the mixed-determined nature of this system which enables determination of the unknown nodal displacements, and therefore the strains, deviatoric stresses and von Mises stress, without calculation of the hydrostatic pressures. This characteristic can be implemented in various ways. For example, the singular value decomposition (SVD) [18] could be used to diagonalize the matrix and identify the equations causing the instability.

An alternative approach, which does not require computation of the SVD, uses a variation of LU decomposition [18] to calculate the nodal displacements while filtering out the calculation of the hydrostatic pressures. The steps in this procedure are as follows:

(i) The columns of matrix $\mathbf{K}^*$ and rows of vector $\mathbf{x}^*$ are placed in reverse order, producing $\mathbf{K}$ and $\mathbf{x}$, respectively. This reverses the order of unknowns in the system of equations and is necessary so that the unknown nodal displacements are calculated first, subsequent to LU decomposition. Unfortunately, this also has the effect of producing a matrix for which pivoting is problematic.

(ii) A least squares system $\mathbf{K}^T\mathbf{K}\mathbf{x} = \mathbf{K}^T\mathbf{b}$ is used to ensure nonzero diagonal entries.

(iii) The matrix $\mathbf{K}^T\mathbf{K}$ is decomposed using LU decomposition, so that

$$\mathbf{K}^T\mathbf{K} = \mathbf{L}\mathbf{U} \tag{2}$$

so that

$$\mathbf{L}\mathbf{y} = \mathbf{K}^T\mathbf{b} \tag{3}$$

where

$$\mathbf{U}\mathbf{x} = \mathbf{y}\ . \tag{4}$$

(iv) Equation (3) is solved for $\mathbf{y}$.

(v) Let m be the number of elements in the finite element mesh and therefore the number of unknown hydrostatic pressures. The system given by Eq. (4) is then reduced by eliminating the first m rows and m columns of matrix $\mathbf{U}$ to produce the reduced matrix $\mathbf{U}_r$, and also by eliminating the first m elements of both $\mathbf{x}$ and $\mathbf{y}$ to produce $\mathbf{x}_r$ and $\mathbf{y}_r$, respectively. This gives the system

$$\mathbf{U}_r\mathbf{x}_r = \mathbf{y}_r \tag{5}$$

which can be solved for $\mathbf{x}_r$, a vector consisting only of nodal displacements.

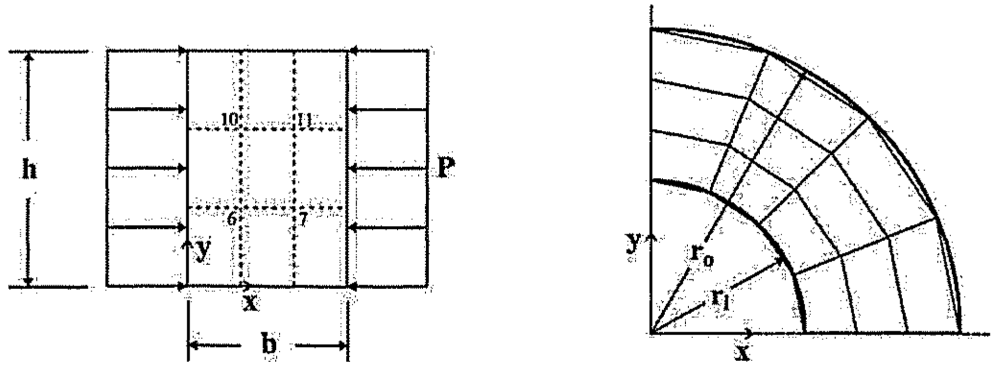
In the next section, two examples are employed to demonstrate that the condition number of $\mathbf{U}_r$ is significantly lower than that of the original stiffness matrix $\mathbf{K}^*$.

EXAMPLES

Numerical results for widely used test cases were produced using a finite element code written by the authors. This finite element formulation uses two-dimensional, isoparametric, quadrilateral elements, each element having four nodes and eight displacement degrees of freedom [19]. Variations in x and y displacements are represented by the usual four-node shape functions, but the pressure in each element is assumed constant. In the literature, this element is referred to as Q4P1 [7] or Q1P0 [20]. For integration, four Gauss points per element are used. For these preliminary tests, a classical Lagrange multiplier technique is used, the relationship between strain and displacement follows the classical linear form, and the relationship between the normal stresses and normal strains is given in terms of stress differences σ_i-σ_j. In the case of incompressibility, the total strain energy is equal to the distortional energy, and is therefore a function of only normal stress differences, σ_i-σ_j, and shearing stresses, τ_{ij}.

Rectangular Block in Plane Strain.

Using the authors' code, the proposed strategy was applied to the analysis of a rectangular block compressed by a uniform pressure acting on two opposite faces as illustrated in Fig. 1(a). The block is constrained from expanding in the z-direction by a pair of rigid walls (not shown in the figure), thereby creating a state of plane strain in the x-y plane.



(a) (b)

Figure 1: (a) Compression of a rectangular block by pressure P; and (b) a plane within a thick-walled cylinder of inner radius r_i and outer radius r_o. The inner and outer radii are subjected to pressures p_i and p_o, respectively. Sample finite element grids are shown.

The analytical solution for this elementary uniform field is given by

$$u = -P\frac{(1-\nu^2)}{E}\left(x - \frac{b}{2}\right) \quad \text{and} \quad v = P\frac{\nu(1+\nu)}{E}\left(y - \frac{h}{2}\right) \tag{6}$$

where u and v are the displacements in the x- and y- directions, respectively, E is Young's modulus and $\nu = 0.5$. For the arbitrarily selected values of $P = 0.1$ kN/mm^2, $E = 70$ kN/mm^2, $b = 100$ mm, and $h = 200$ mm, Eqs. (6) were used to produce displacement boundary conditions so that the problem could be recast as a Dirichlet boundary value problem (where, on the boundary, u and v are known but P is unknown). Plane strain problems such as this are common in the patch test, and are used here since they exhibit the same numerical instability as in three-dimensional cases.

Using either the standard u-p approach or the modified strategy proposed, the calculated nodal displacements were correct within round off. The issue here, however, is the stability of the calculations. The condition numbers for **K***, **U** and $\mathbf{U_r}$ are given in Table 1. For each matrix, the condition number was calculated as the ratio of the largest to smallest singular value in the SVD. Recall that in the modified approach, the unknown nodal displacements are calculated using $\mathbf{U_r}$, which is constructed from matrix **U** by eliminating the rows and columns corresponding to the hydrostatic pressures. Clearly, matrix **U** manifests the instability inherent in the original stiffness matrix **K***, while the reduction to $\mathbf{U_r}$ achieves a dramatic decrease in the condition number, and therefore a significant increase in numerical stability. Without the proposed strategy, or its equivalent, the unstable calculation of hydrostatic pressures may cause the solution process to stop prior to calculation of the nodal displacements.

Table 1. Condition numbers for matrices used in calculating displacements

Matrix	Rectangular Block	10×10 Cylinder	15×15 Cylinder	20×20 Cylinder
K*	1.07×10^{17}	1.92×10^{16}	5.71×10^{16}	1.15×10^{17}
U	2.51×10^{16}	1.09×10^{17}	3.42×10^{17}	1.90×10^{18}
$\mathbf{U_r}$	6.81	252	727	1548

Thick-walled Cylindrical Pressure Vessel

Consider a thick-walled cylindrical pressure vessel of inner radius, r_i, outer radius, r_o, and subjected to internal pressure, p_i, and external pressure, p_o. This is the classical Lamé problem [19]. See Fig. 1(b). If the cylinder is axially restrained (i.e., in plane strain) and is made of a linearly elastic, incompressible material, the analytical solution for displacements in the x- and y- directions is given by

$$u = \frac{3(r_i r_o)^2 (p_i - p_o)}{2\,E\,r\left(r_o^{\,2} - r_i^{\,2}\right)} \cos\theta \quad \text{and} \quad v = \frac{3(r_i r_o)^2 (p_i - p_o)}{2\,E\,r\left(r_o^{\,2} - r_i^{\,2}\right)} \sin\theta \tag{7}$$

where r and θ are the polar coordinates of a point, easily written as x- and y- coordinates. For the arbitrarily selected values of $r_i = 100$ mm, $r_o = 200$ mm, $p_i = 0.2$ kN/mm^2 and $p_o = 0.1$ kN/mm^2, Eqs. (7) were used to produce displacement boundary conditions in recasting the problem as a Dirichlet boundary value problem. Results were produced for three grids (given as number of elements in the radial direction × the number of elements in the circumferential direction): 10×10, 15×15, and 20×20. The average (maximum) percent difference between the von Mises stress at the natural centers of the elements and the values obtained using the finite element code for these grids are 0.491%(0.550%), 0.219%(0.245%), and 0.123%(0.138%), respectively. The condition numbers of **K***, **U**, and $\mathbf{U}_r$ for each of the grids is given in Table 1.

Again, dramatic reductions in condition number are achieved by eliminating the rows and columns corresponding to the hydrostatic pressures, producing $\mathbf{U}_r$ which is used to determine the nodal displacements. All of the condition numbers increase with the grid size. This is expected, however, when going from a 10×10 grid to a 20×20, since the size of the stiffness matrix **K*** increases from 262×262 to 1122×1122.

For both this example, and the rectangular block, the count condition is either violated or low compared to the recommended ratio [17]. It is possible, however, to selectively eliminate those unknowns which are indeterminate thereby producing a reduced, nonsingular system of equations in terms of the unknown nodal displacements. This is especially significant since the examples use the Q4P1 element which has been criticized for violating the count condition but is still widely used [7].

Although the reduced systems are nonsingular and stable, the calculations involved in the LU decomposition may involve steps, such as division by a small number, which could affect the numerical stability of the overall solution process. The extent of this instability would depend on the specific nature of the matrix **K** and warrants further study.

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