ABSTRACT
The main objective of this paper is to present a theoretical approach to model the vibro-acoustic behavior of flat sandwich composite panels. Two models are studied: symmetrical laminate composite and sandwich composite panel. The theories are developed in a wave approach context. It is shown that a discrete layers sandwich composite panel modeling type leads to a 12th order relation of dispersion while a laminate composite panel modeling leads to a 6th order relation of dispersion. The two models give similar results at low frequencies but the modeling of a sandwich panel using the laminate panel theory leads to inaccuracies at high frequencies. The dispersion relations are first solved in the context of generalized polynomial complex eigenvalues problems. Next, the dispersion relations are used to derive the analytical expression of the critical frequencies and to calculate the natural frequencies of the panel. Using the dispersion relation’s solutions, the study is then focused on the numerical computation of the group velocity, the modal density and the total transmission loss.

Analyse du comportement vibro-acoustique des panneaux plans composites sandwich

RESUME
L'objectif principal de ce travail est le développement d'une approche théorique pour la modélisation du comportement vibro-acoustique des panneaux plans, sandwich composites. Deux modèles sont étudiés: panneau stratifié symétrique composite et sandwich composite. Les théories sont développées dans un contexte d'approche d'onde. Il est montré dans cet article qu'un panneau sandwich composite modélisé par une approche de couches discrètes a une relation de dispersion d'ordre 12 tandis que la modélisation de type panneau composite stratifié symétrique mène à une relation de dispersion d'ordre 6. Les deux modèles donnent des résultats semblables en basses fréquences mais il est montré que la modélisation d'un panneau sandwich en utilisant une approche de panneau stratifié est erronée en hautes fréquences. Dans ce papier, les relations de dispersion sont résolues dans un contexte de problème aux valeurs propres polynomial complexe. Les relations de dispersion sont ensuite utilisées pour développer des expressions analytiques des fréquences critiques et pour calculer les fréquences naturelles du panneau. En utilisant les solutions de la relation de dispersion, le papier présente des exemples de calculs de la vitesse de groupe, de la densité modale et du coefficient d'affaiblissement.
1. INTRODUCTION

Sandwich composite panels are widely used in aerospace, aeronautical and automotive industries. Such panels are made up of thin composite face sheets and a shearing core. The core is generally made up of a softer material than the skins but the whole panel is characterized by an important strength and low total weight. The vibro-acoustic modeling of sandwich panels is investigated in a large number of papers. The first expression of the modal density for isotropic sandwich panels is proposed by Wilkinson [1]. One year later, Erickson [2] studied the effect of the anisotropy of the core on the modal density. His approach accounts for the skin's bending and the panel's rotational inertia while the core is described by an equivalent shear modulus, computed as the geometric average of the shear modulus along x and y directions. The bending stiffness of the core is neglected. The theories suggested by these two authors ([1], [2]) are compared to experiments by Clarkson [3]. As a logical continuity, Renji et al. [4] propose a model for the modal density of orthotropic sandwich panels. The core is characterized by an equivalent transverse shearing (geometric average) and the rotational inertia terms are neglected. In a more recent study, Nilsson et al. [5] consider the problem of sandwich orthotropic structures. A sixth order relation of dispersion is proposed for sandwich beams. The beam's rotational inertia and bending as well as the core's transversal shearing are considered. The principle of Hamilton is used to model the beam. The strain energy of the beam is defined as the sum of potential energies due to the pure bending of the panel, the pure bending of the skins and the transversal shearing of the core. The theories referred to above propose analytical solutions for the relation of dispersion. Authors [1-5] show that various simplifying assumptions are necessary to solve analytically the dispersion problem. It is concluded in a recent paper [6] that the general dispersion problem of flat laminated composite panels does not have an analytical solution. A wave approach numerical method is proposed by Ghinet and Atalla [6] to solve the relation of dispersion of flat laminated composite panels. The relation of dispersion is written in the form of a polynomial generalized complex eigenvalues problem. The Mindlin type displacements field is used for each layer which allow for bending and transversal shearing. The layers' physical properties are smeared through the thickness of the panel according to Berthelot's [7] description of laminate composite structures. Moreover, the rotational inertia is accounted for and shear correction factors are calculated explicitly following the approach described by Batoz and Dhatt [16]. The governing system of dynamic equilibrium equations is written according to a hybrid vector composed by the displacements-rotations field and the resulting forces and moments of the panel. The solutions are used [6] to calculate the group velocity, the modal density and the radiation efficiency. The non-resonant transmission coefficient is calculated according to Lesueur [8] classical approach and is corrected by a spatial windowing of the radiating field method described in reference [9]. The modal density, the radiation efficiency and the non-resonant and resonant transmission coefficients are used in a SEA framework to estimate the total transmission loss of the laminate composite structures. This approach was successfully employed for the acoustic design of curved sandwich composite panels with noise control treatments [10, 11]. In the context of predictive SEA, a wave approach for curved sandwich panels was proposed by K. Heron [12]. The dispersion relation of such panels is shown to have two propagating solutions at low frequencies and five propagating wave solutions at high frequencies. The problem's dimension is 47 by 47 and is written according to a hybrid vector composed by the panel's displacements field, the resultant forces and moments as well as the interlayer forces. In the context of the laminate composite cylinders modeling, two models were presented and compared in reference [13]: symmetrical laminate composite and discrete thick laminate composite. The latter was shown to handle accurately, as a particular case, sandwich composite shells [13, 14]. In the two presented models, membrane, bending, transversal shearing as well as rotational inertia effects and orthotropic angle-ply of the layers were considered. As an example [14], a curved sandwich composite panel has 21 equations and 21 unknown variables but the polynomial complex eigenvalues problem representing the dispersion relation is of the 42nd order. The symmetrical laminate composite and discrete thick laminate composite curved panel models could be used for flat panels modeling by setting a large curvature radius.
This paper describes the vibro-acoustic modeling of finite sandwich composite panels. The proposed model accounts for orthotropy and has a 12th order relation of dispersion. It is designed for complex fast numerical applications such as: acoustic design and optimization as well as inverse characterization methods to evaluate the panel’s physical properties. The present model uses a discrete displacement field for each layer and allows for out of plane displacements and shearing rotations. This displacement field and the discrete layer nature of the theory are adapted to finely model the physical phenomena appearing at high frequencies where the difference of stiffness between the skins and the core allows for the separate bending motion of the skins. Each discrete layer is considered laminate (composite) but the physical properties are smeared through the thickness of each layer so that the problem’s dimension remains unchanged. The solutions of the dispersion relation are used here to compute the group velocity, the modal density and the total transmission loss. Additionally, a symmetrical laminate composite modeling type is presented and used here to validate and demonstrate the accuracy of the sandwich type modeling. Moreover, isotropic laminate panel solutions are symbolically developed and used to analyze the vibro-acoustic asymptotic behaviors. The sandwich composite panel modeling is finally compared to existing models in the literature, finite elements results and experimental data to demonstrate its accuracy.

2. SYMMETRICAL LAMINATE COMPOSITE PANEL

2.1. GEOMETRY

The present study is dedicated to laminated composite flat panels modeling. In Figure 1 is represented the geometrical configuration of the composite panel, of side sizes $L_x$ and $L_y$ and total thickness $h$. The layered constitution is considered symmetrical. The origin of the $z$ axis is defined on a reference surface passing through the middle thickness of the panel.

![Figure 1. Dimensions of the panel.](image)

2.2. DISPLACEMENT FIELD AND GOVERNING EQUATIONS

For any point belonging to the symmetrically laminated composite panel, the displacement field is defined by the Mindlin model:

$$
\begin{align*}
    u(x,y,z) &= u_0(x,y) + zj_x(x,y) \\
    v(x,y,z) &= v_0(x,y) + zj_y(x,y) \\
    w(x,y,z) &= w_0(x,y)
\end{align*}
$$

Laminas’ properties are smeared through the panel’s thickness. The bending moments $(M_x, M_y, M_{xy})$ and the transversal shear forces $(Q_x, Q_y)$ are defined in the appendix. The governing differential equations
of the symmetrical laminate composite panel are written using the usual notations [7]:

\[ Q_{xx} + Q_{yy} = m s w,1t \]
\[ M_{xx} + M_{xy} - Q_x = I_j x,1t \]
\[ M_{xy} + M_{yy} - Q_y = I_j y,1t \]

with \( m_s \) the mass per unit area, \( I_z \) the rotational inertia computed using relation (A3) in the appendix, and where the bending stiffness \( D_{ij} \) and shear stiffness \( F_{ij} \) are defined as:

\[ D_{ij} = \sum_{k=1}^{N} Q_i^k \frac{h_k^3 - h_k^2}{3}; \quad F_{ij} = k_{ij} \sum_{k=1}^{N} C_{ij}^k \left( h_k^3 - h_k^2 \right); \]

with \( k_{ij} \) the shear correcting factors and \( Q_i^k, C_{ij}^k \) the general elastic constants of the symmetrical laminate composite panel defined by the relations (A4) and (A5) of the appendix. The elastic constants of each lamina are represented according to the orthotropic angle \( J_k \) \((k=1,2,3)\) shown in Figure 2 and defined as the angle between the reference co-ordinate system of each lamina \( k \) \((L-O-T)\) and the global co-ordinate system of the panel \((x-0-y)\).

\[ \text{Figure 2. Orthotropic direction of a lamina.} \]

2.3. DISPERSION RELATIONS

This section is dedicated to the theoretical study of the dispersion solutions and their asymptotical behaviors. The first part of the study is concerned with the laminated isotropic panels' theoretical development which is a simplification of the main theory. It allows the symbolical analysis of the dispersion relations and a fast development of the asymptotical tendencies. The second part of this section is concerned with the development of the general relation of dispersion for laminated composite panels.

a. Dispersion relation of thick laminated isotropic panels

The case of thick laminate isotropic flat panels is studied in this section. The system of governing differential equations (2) is recast using the following simplifications: \( D_{11}=D_{22}=D; \quad D_{12}=vD; \quad D_{66}=(1-v)D; \quad F_{44}=F_{55}=Gh \) and the problem is solved analytically. It leads to the following dispersion relation:
\[
\frac{1}{2} [k^3 D(\nu - 1) + 2(I, \omega^3 - Gh)] \times \\
\times [I, m_s \omega^4 - Ghm_s \omega^3 - k^3 (GhI_s + m_s D) \omega^3 + k^4 GhD] = 0
\] (4)

In the above equation, the following notations are used:

\[
D = \sum_{k=1}^{N} \frac{E_k}{\nu_k (1 - \nu_k)} \frac{h_{sk}^3 - h_k^3}{3} \quad \text{and} \quad Gh = \sum_{k=1}^{N} G_k (h_{sk} - h_k)
\]

where \(h_{sk}\) and \(h_k\) are defined by relations (A2). The mass per unit area \(m_s\) and the rotational inertia \(I_s\) are computed using relations (A3), \(E_k\) and \(G_k\) are the Young modulus and respectively the shear modulus of any lamina \(k\), while \(N\) is the total number of lamina composing the panel.

Because of the layers' isotropy, it is observed that relation (4) does not depend on the heading direction. This 6th order dispersion relation has three conjugate solutions. The first term in relation (4) leads to an evanescent wave solution expressed as follows:

\[
k = \pm \sqrt{-2 \frac{I, \omega^3 - Gh}{D(\nu - 1)}},
\] (5)

while the second term leads to a group of four solutions:

\[
k = \pm \sqrt{\frac{1}{2} \frac{\omega (GhI_s + m_s D) \pm \sqrt{\omega^3 (GhI_s - m_s D)^3 + 4(Gh)^3 D m_s}}{GhD}},
\] (6)

where the propagating wave solutions are given by the following expression:

\[
k = \pm \sqrt{\frac{1}{2} \frac{\omega (GhI_s + m_s D) + \sqrt{\omega^3 (GhI_s - m_s D)^3 + 4(Gh)^3 D m_s}}{GhD}}.
\] (7)

The second term of relation (4) is rewritten as:

\[
k^4 D - m_s \omega^3 + \left( \frac{I, m_s}{Gh} \omega^3 - k^3 \frac{m_s D}{Gh} \right) \omega^3 - k^4 I_s \omega^3 = 0.
\] (8)

In relation (8), it is now easy to observe three tendencies:

- a first term: \(k^4 D - m_s \omega^3\) corresponding to bending behaviors (thin panel);
- a second term: \(\left( \frac{I, m_s}{Gh} \omega^3 - k^3 \frac{m_s D}{Gh} \right) \omega^3\) corresponding to shear effect;
- a third term: \(-k^3 I_s \omega^3\) corresponding to rotational inertia behavior.

The asymptotical development of relation (8) for \(\omega \to 0\) and \(\omega \to \infty\) leads to the dispersion relations associated to bending and shear, respectively:
The dispersion curve (propagating wave solution) for the thick isotropic panel defined in Table 1 is represented in Figure 3. Also, the asymptotes for bending (9) and shear (10) behaviors are plotted. It is observed in Figure 3 that the panel has pure bending behaviors at low frequencies and pure shearing behaviors at high frequencies. This remark stresses the fact that the laminate approach will fail for sandwich panels since at high frequencies the system’s dynamics is instead controlled by bending effects in the skins.

\[ k_{\text{bending}} = \sqrt{\frac{\omega}{D}} \text{ for } \omega \to 0 \]  
\[ k_{\text{shear}} = \omega \sqrt{\frac{4I_s m_s}{I_s G h + m_s D}} \text{ for } \omega \to \infty \]

The dispersion curve (propagating wave solution) for the thick isotropic panel defined in Table 1 is represented in Figure 3. Also, the asymptotes for bending (9) and shear (10) behaviors are plotted. It is observed in Figure 3 that the panel has pure bending behaviors at low frequencies and pure shearing behaviors at high frequencies. This remark stresses the fact that the laminate approach will fail for sandwich panels since at high frequencies the system's dynamics is instead controlled by bending effects in the skins.

![Dispersion curves - thick isotropic panel](image)

**Figure 3. Dispersion curves for a thick aluminium plate.**

<table>
<thead>
<tr>
<th>Aluminium panel:</th>
<th>$L_x=2.45$ m; $L_y=1.22$ m; $h=10$ cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Young modulus</td>
<td>$E$ (Pa) $7.2 \times 10^{10}$</td>
</tr>
<tr>
<td>Mass density</td>
<td>$\rho$ (kg/m$^3$) 2780</td>
</tr>
<tr>
<td>Poisson's ratio</td>
<td>$\nu$ 0.33</td>
</tr>
<tr>
<td>Damping</td>
<td>$\eta$ 0.007</td>
</tr>
</tbody>
</table>

Table 1. The physical properties of the aluminium panel.

b. Dispersion relation for symmetrically laminated composite panels

To account for orthotropy, the governing differential equations of the composite (2) are rewritten as described in Ref. [17] (Chapter 2) and expressed in terms of a displacement–rotation vector $(e)^t = (w, \varphi_x, \varphi_y)$; here index $t$ refers to the transpose operator. Assuming the problem’s solution in the
form \( \langle e \rangle = \langle e \rangle \exp (jk_x x + jk_y y - j\omega t) \), the system is expressed as a generalized polynomial complex eigenvalue problem:

\[
k_p^2 [A_1] \{ e \} - i k_p [A_2] \{ e \} - [A_0] \{ e \} = 0;
\]

\[\text{(11)}\]

where \( i = \sqrt{-1} \) and \([A_0], [A_1], [A_2]\) are matrices of dimension \( 3 \times 3 \) defined as:

\[
[A_0] = \begin{bmatrix}
m_0 \omega^2 & 0 & 0 \\
0 & -F_{s2} + \omega^2 I_x & -F_{s3} \\
0 & -F_{s4} & -F_{s5} + \omega^2 I_x
\end{bmatrix};
\]

\[
[A_1] = \begin{bmatrix}
\alpha_{12} & 0 & 0 \\
0 & -\alpha_{13} & 0 \\
\alpha_{23} & 0 & 0
\end{bmatrix};
\]

\[
[A_2] = \begin{bmatrix}
0 & \beta_{22} & \beta_{23} \\
0 & \beta_{23} & \beta_{23}
\end{bmatrix};
\]

\[\text{(12)}\]

with \( \alpha_{12}, \alpha_{13}, \beta_{12}, \beta_{13}, \beta_{22}, \beta_{23} \) given in Ref. [17] (Chapter 2).

Assuming \( \lambda = i \cdot k_p \) in (11), the system has 6 complex conjugate eigenvalues propagating or evanescent in two opposite directions. Relation (11) represents the 6th order dispersion relation of the laminate composite panels.

### 2.4. MODAL DENSITY

The modal density is defined as the number of resonant structural modes within a studied frequency band divided by the frequency bandwidth. The angular distribution of the modal density is classically expressed in terms of the ratio of the structural wave-number to the group velocity [8]:

\[
n(\varphi, \omega) = \frac{A}{2\pi^2} \left| \frac{k(\varphi, \omega)}{c_g(\varphi, \omega)} \right|.
\]

\[\text{(13)}\]

The modal density is obtained numerically by integrating relation (13) over heading angles \( n(\omega) = \int_0^{2\pi} n(\varphi, \omega) d\varphi \), while the structural wave number \( k(\varphi, \omega) \) and the group velocity \( c_g(\omega, \varphi) = \omega / dk \) are computed numerically from the solution of the dispersion relation (11).

**Case of thick laminated isotropic panels**

Using the dispersion relation (8) for thick isotropic panels and the definition of the group velocity, the following relation is obtained:

\[
c_g = \frac{k}{\omega} \frac{2k^2 D - \omega^2 \left( I_z + \frac{m_D}{Gh} \right)}{m_s \left( 1 - 2 \frac{I_z - \omega^2}{Gh} \right) + k^2 \left( I_z + \frac{m_D}{Gh} \right)}.
\]

\[\text{(14)}\]

The group velocity's bending and shear tendencies are expressed using the asymptotical relations (9) and (10) as follows:

\[
c_{g, \text{bending}} = 2k \frac{D}{m_s} \quad \text{for} \quad \omega \to 0; \quad c_{g, \text{shear}} = \frac{I_z Gh + m_D}{4I_z m_s} \quad \text{for} \quad \omega \to \infty
\]

\[\text{(15)}\]
The group velocity curves (Eq. (14)) of the thick isotropic panel described in Table 1 as well as the asymptotical tendencies for bending and shear behaviors (15) are represented in Figure 4. Its modal density and the asymptotical tendencies for bending and shear behaviors are represented in Figure 5.

![Figure 4. The group velocity of a thick aluminum plate.](image)

![Figure 5. The modal density of a thick aluminum plate.](image)
sandwich composite panel

This section considers the special case of sandwich composite panels. The theory described herein has the overall dimension of 12 by 12 and is based on a discrete displacement field approach. It allows for a correct description of the vibro-acoustic behaviors over the whole frequency range.

2.5. ASSUMPTIONS AND GEOMETRY

The present model is based on the following assumptions:
- the thickness of the core is higher than that of the skins;
- the core contributes only by transversal shear stresses;
- transversal shear stresses are neglected in the skins;
- the core and the skins are assumed incompressible through the thickness;

In Figure 6a are represented the geometrical characteristics of a sandwich composite panel of side sizes $L_x$ and $L_y$ and thickness $h$. For any point $M$ belonging to the core, the Mindlin-type displacement field is defined as represented in Figure 6b.

![Figure 6. Dimensions of the panel and the Mindlin-type displacements field of the core.](image)

2.6. DISPLACEMENTS FIELDS

The displacements fields of the skins are defined so as to obey the sandwich panels' assumptions. The skins are thinner than the core and act by bending behaviors. Their displacement field is built using the Love-Kirchhoff's assumptions but is corrected to account for the rotational influence of the transversal shearing in the core. This correction appears in the form of the constants $c_{1x}$, $c_{3x}$, and $c_{1y}$, $c_{3y}$ as stated in the following relations:

\[
\begin{align*}
  u_1 &= u_{01} - z \frac{\partial w}{\partial x} + c_{1x} \\
  v_1 &= v_{01} - z \frac{\partial w}{\partial y} + c_{1y} \\
  w_1 &= w \\
  u_2 &= u_{02} - z \frac{\partial w}{\partial x} + c_{3x} \\
  v_2 &= v_{02} - z \frac{\partial w}{\partial y} + c_{3y} \\
  w_2 &= w
\end{align*}
\]
The Mindlin model is used to describe the displacement field of the core. The rotation effects of the transversal shearing in the core as well as the bending of the panel are described by the rotations angles $\varphi_x, \varphi_y$ and the transversal displacement $w$ as:

$$
\begin{align*}
u_1(x,y,z) &= v_{01}(x,y) + z\varphi_x(x,y) \\
v_2(x,y,z) &= v_{02}(x,y) + z\varphi_y(x,y) \\
w_1(x,y,z) &= w(x,y)
\end{align*}
$$

(17)

Because of the assumed perfect bounding of the layers, the displacement field remains continuous throughout the interface between two consecutive layers. To enforce this continuity the following conditions are written:

$$
\begin{align*}
u_1(x,y,h_2/2) &= \nu_2(x,y,h_2/2); \\
v_1(x,y,h_2/2) &= \nu_2(x,y,-h_2/2); \\
u_1(x,y,h_2/2) &= \nu_2(x,y,-h_2/2); \\

\end{align*}
$$

(18)

In the case of flat panel dynamics, the bending and transversal shearing are decoupled from the membrane behaviors. Consequently, the membrane-type displacements of the layers' neutral plan are not considered. Moreover, the out-of-plane compression stresses through the panel's thickness are considered constant. Using these assumptions, the system of continuity equations (18) has the solutions $c_{1x}, c_{3x}, c_{1y}, c_{3y}$:

$$
\begin{align*}
c_{1x} &= \frac{h_2}{2} \frac{\partial w}{\partial x} + \varphi_x \\
c_{3x} &= -\frac{h_2}{2} \frac{\partial w}{\partial x} + \varphi_x \\
c_{1y} &= \frac{h_2}{2} \frac{\partial w}{\partial y} + \varphi_y \\
c_{3y} &= -\frac{h_2}{2} \frac{\partial w}{\partial y} + \varphi_y
\end{align*}
$$

(19)

Using relations (16), (17) and (19), the displacements fields of each of the three layers are written as follows:

$$
\begin{align*}
u_1 &= -z \frac{\partial w}{\partial x} + \frac{h_2}{2} \psi_x \\
v_2 &= -z \frac{\partial w}{\partial x} + z \psi_x \\
v_3 &= -z \frac{\partial w}{\partial x} + \frac{h_2}{2} \psi_x \\

\end{align*}
$$

$$
\begin{align*}
u_1 &= -z \frac{\partial w}{\partial y} + \frac{h_2}{2} \psi_y \\
v_2 &= -z \frac{\partial w}{\partial y} + z \psi_y \\
v_3 &= -z \frac{\partial w}{\partial y} + \frac{h_2}{2} \psi_y
\end{align*}
$$

(20)

where the following notations are used:

$$
\psi_x = \frac{\partial w}{\partial x} + \varphi_x; \quad \psi_y = \frac{\partial w}{\partial y} + \varphi_y.
$$

(21)

2.7. INTERLAYER STRESSES CONTINUITY RELATIONS

Relations of stresses' dynamic equilibrium are written for each layer separately in order to develop the continuity of stresses relations at the interface between the layers. These relations are next integrated through the layer's thickness and the dynamic equilibrium relations along $x$ and $y$ directions are obtained. For the top skin – core interface, the relations of continuity are written as follows:
with:

\[
\tau_{zz}^{(1)} = \tau_{zz}^{(2)} = C_{zz}^{(2)} \psi_z + C_{zz}^{(3)} \psi_y \\
\tau_{zy}^{(1)} = \tau_{zy}^{(3)} = C_{zy}^{(2)} \psi_z + C_{zy}^{(4)} \psi_y
\]

Using the notations:

\[
A_y = \frac{h_1 h_2}{2} Q_y^{(i)}; \\
B_y = \frac{h_1}{2} \frac{h_1 + h_2}{2} Q_y^{(1)}
\]

with the elastic constants of the skins \( Q_{yy}^{(1)} = Q_{yy}^{(3)} \), defined by the relations (A4) and the elastic constants of transversal shearing of the core \( C_{yy}^{(3)} \) defined by the relations (A5).

The development of the stresses’ continuity relations between the core and the bottom skin is identical to (22) and identical relations of continuity are obtained (see Ref. [17]).

2.8. DYNAMIC EQUILIBRIUM RELATIONS

The panel’s dynamic behaviors are governed by the dynamic equilibrium relations of the forces and moments along \( x, y \) and \( z \) directions. The sandwich-type panel assumptions are considered; the membrane forces \( N_x, N_y \) and \( N_{xy} \) and the bending moments \( M_x, M_y \) and \( M_{xy} \) are computed through the thickness of the skins while the transversal shearing forces \( Q_x \) and \( Q_y \) are expressed through the core’s thickness. Considering the stresses’ continuity relations along \( x, y, \) and \( z \) directions as well as the panel’s incompressibility along the \( z \) direction and integrating through the panel thickness, the following equilibrium relations are written:

\[
Q_{xx} + Q_{yy} = m_x w_{,tt} \\
M_{xx} + M_{xy} - Q_y = I_x u_{,tt} \tag{25}
\]

The transversal shearing forces \( Q_x \) and \( Q_y \) are expressed from the last two relations of system (25). These forces are replaced in the first fundamental equation to express the following relation of motion:

\[
M_{xx} + 2M_{xy} + M_{yy} - (I_x u_{,tt})_x - (I_y u_{,tt})_y + m_x \omega^2 w = 0 \tag{26}
\]

with the rotational inertia terms computed as follows:

\[
I_x u_{,tt} = \omega^2 \left[ (\rho_1 H_1 + \rho_2 H_2) w_{,s} - (\rho_1 H_2 + \rho_2 H_3) \psi_z \right] \\
I_y u_{,tt} = \omega^2 \left[ (\rho_1 H_1 + \rho_2 H_3) w_{,y} - (\rho_1 H_2 + \rho_2 H_3) \psi_y \right]
\]

and where the following notations are used:
\[
H_1 = \frac{3h_1h_2^3 + 6h_1^2h_3 + 4h_1h_3^2}{6}; \quad H_2 = \frac{h_1h_2(h_1 + h_3)}{2}; \quad H_3 = \frac{h_1^3}{12}.
\]

The relation (26) and the stresses' continuity relations given in Ref. [17] compose the system of dynamic equilibrium equations of sandwich composite panels. This system is expressed in a matrix form using a displacements vector \( \mathbf{e} = \{w, \psi_x, \psi_y\}^T \) and an assumed solution of the form
\[
\mathbf{e} = \{\mathbf{e}\} \exp(jk_x x + jk_y y - j\omega t),
\]
where \( k_x \) and \( k_y \) are the components of the structural wave number \( k \) defined as:
\[
\begin{align*}
k_x &= k \cos \varphi \\
k_y &= k \sin \varphi
\end{align*}
\]
and \( \varphi \) is the heading direction.

The system of dynamic equilibrium equations of the panel is expressed in the following matrix form:
\[
(k^4 [A] + jk^3 [B] + k^2 [C] + jk [D] + [E]) \{\mathbf{e}\} = [0] \{\mathbf{e}\}
\]

where \([A], [B], [C], [D]\) and \([E]\) are 3\(\times\)3 real matrices defined as follows:
\[
[A] = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}; \quad [B] = \begin{bmatrix}
0 & b_{12} & b_{13} \\
b_{21} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}; \quad [C] = \begin{bmatrix}
c_{11} & 0 & 0 \\
0 & c_{22} & c_{23} \\
0 & c_{32} & c_{33}
\end{bmatrix};
\]
\[
[D] = \begin{bmatrix}
d_{11} & d_{12} & d_{13} \\
d_{21} & 0 & 0 \\
d_{31} & 0 & 0
\end{bmatrix}; \quad [E] = \begin{bmatrix}
e_{11} & 0 & 0 \\
0 & e_{22} & e_{23} \\
0 & e_{32} & e_{33}
\end{bmatrix};
\]

where the matrices' elements, \(a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta}, e_{\alpha\beta}, \alpha, \beta = 1, 2, 3\) can be easily determined as detailed in Ref. 17.

Considering \( \lambda = \pm jk \), relation (28) can be expressed in the form:
\[
(\lambda^4 [A] - \lambda^3 [B] - \lambda^2 [C] + \lambda [D] + [E]) \{\mathbf{e}\} = [0] \{\mathbf{e}\}.
\]

System (29) is a complex polynomial eigenvalues problem of fourth order and can be transformed as follows:
\[
\begin{bmatrix}
[J] & [0] & [0] & [0] \\
[0] & [J] & [0] & [0] \\
[0] & [0] & [J] & [0]
\end{bmatrix}
\begin{bmatrix}
\lambda & \lambda^2e & \lambda^3e & \lambda^4e
\end{bmatrix}
= \begin{bmatrix}
[-E] & [0] & [0] & [0] \\
[0] & [J] & [0] & [0] \\
[0] & [0] & [J] & [0] \\
[0] & [0] & [0] & [J]
\end{bmatrix}
\begin{bmatrix}
e \\
\lambda e \\
\lambda^2e \\
\lambda^3e
\end{bmatrix}
\]

(30)
to obtain a generalized complex eigenvalues problem. In relation (30), \([J]\) is the identity matrix and \([0]\) is the zero matrix. This problem has 12 conjugated complex eigenvalues corresponding to propagative and
evanescent waves in two opposite directions. Equation (30) represents the dispersion relation of the sandwich composite panels.

2.9. CRITICAL FREQUENCIES

The critical frequency region of a sandwich composite panel is given by the particular solution of the dispersion equation (28) at coincidence, when the structural wave number \( k \) matches the acoustic wave number \( k_0 \): 

\[
\begin{align*}
&k^4 [A] \{ e \} + jk^3 [B] \{ e \} + k^2 [C] \{ e \} + jk [D] \{ e \} + [E] \{ e \} = [0] \{ e \} \\
&k_0 = k = \omega / c_0
\end{align*}
\]

Equation (30) represents the dispersion relation of the sandwich composite panels.

For a given heading \( \phi \), the critical frequency is computed numerically from system (31). The following expansions: 
\[
[C] = \omega^2 [C_1] + [C_2] \quad \text{and} \quad [E] = \omega^2 [E_1] + [E_2]
\]
are used to explicitly depict the frequency dependency and write the problem in the following form:

\[
\begin{align*}
\omega^2 \left[ \frac{[A]}{c_0^2} + \frac{[C_1]}{c_0^2} \right] \{ e \} + j\omega^2 \left[ \frac{[B]}{c_0^2} + \frac{[D]}{c_0} \right] \{ e \} + \omega^2 \left[ \frac{[C_2]}{c_0^2} + [E_1] \right] \{ e \} + [E_2] \{ e \} = 0;
\end{align*}
\]

which is a fourth order polynomial eigenvalues problem. Assuming \( \lambda = i \omega \), Eq. (32) can be expressed in the form:

\[
\begin{align*}
\lambda \left[ \begin{array}{c}
[0] - \frac{[C_1]}{c_0^2} + [E_1] \\
[0] - \frac{[B]}{c_0^2} + [D] \\
[0] - \frac{[A]}{c_0^2} + \frac{[C_1]}{c_0^2}
\end{array} \right] \{ e \} = \left[ \begin{array}{c}
-E_2 \\
0 \\
0 \\
0 \\
\lambda^2 e
\end{array} \right];
\end{align*}
\]

(33)

to obtain a generalized eigenvalue problem. This problem has 12 complex conjugate eigenvalues. The critical frequency corresponds to a solution which satisfies the condition \( \lambda = \pm i \omega \), purely imaginary. The critical frequency of the sandwich composite panel is written as:

\[
f_c (\phi) = \frac{\pm i \lambda_c (\phi)}{2 \pi}.
\]

and the limits of the critical frequency region are defined by \( f_c = f_c (\phi = 0) \) and \( f_c = f_c (\phi = \pi / 2) \). For an isotropic panel, the dependency on heading disappears and the critical frequency region reduces to a single critical frequency.

2.10. NATURAL FREQUENCIES

The presented dispersion system can also be used to compute the natural frequencies of the panel using the proper selection of the wavenumber components. For instance, for a simply supported panel of side dimensions \( L_x \) and \( L_y \), the wavenumber components are defined as \( k_x = m \pi / L_x \), \( k_y = n \pi / L_y \). The natural
frequencies associated to each mode \((m, n)\) are obtained by solving the dispersion relation (28), recast in the following form:

\[
(k_m^2 [A] + j k_m^2 [B] + k_m^2 [C_2] + [E_2]) (e) + \omega_m^2 (k_m^2 [C_1] + j k_m^2 [D] + [E_1]) (e) = 0; \quad (35)
\]

where, \(k_m = (k_x^2 + k_y^2)^{1/2}\), and the matrices \([C_1], [C_2], [E_1], [E_2]\) are defined by the relations:

\[
[C] = \omega^2 [C_1] + [C_2] \quad \text{and} \quad [E] = \omega^2 [E_1] + [E_2].
\]

3. NUMERICAL RESULTS

In this section, dispersion curves and results of the group velocity, modal density and transmission loss applied to flat composite laminate and sandwich panels are presented. The two presented modeling approaches are compared: symmetrical laminate composite and sandwich composite panels. The properties of the materials used in this study are presented in Table 2. The panel’s side dimensions are 1.37x1.65m². The core’s thickness is 0.0127m while the skins’ thicknesses are 0.0012m.

<table>
<thead>
<tr>
<th></th>
<th>Graphite/Epoxy Skins</th>
<th>Rigid foam Core</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_L) (Pa)</td>
<td>0.48x10¹¹</td>
<td>0.144x10⁹</td>
</tr>
<tr>
<td>(E_T) (Pa)</td>
<td>0.48x10¹¹</td>
<td>0.144x10⁹</td>
</tr>
<tr>
<td>(G_L) (Pa)</td>
<td>0.181x10¹¹</td>
<td>0.5x10⁸</td>
</tr>
<tr>
<td>(G_{LT}) (Pa)</td>
<td>0.275x10¹⁰</td>
<td>0.5x10⁸</td>
</tr>
<tr>
<td>(G_{TT}) (Pa)</td>
<td>0.275x10¹⁰</td>
<td>0.5x10⁸</td>
</tr>
<tr>
<td>(\nu_{LT})</td>
<td>0.3</td>
<td>0.45</td>
</tr>
<tr>
<td>(\rho) (kg/m³)</td>
<td>1550</td>
<td>110.44</td>
</tr>
</tbody>
</table>

Table 2. The physical properties of the materials used in the sandwich composite panel.

3.1. Dispersion curves

The dispersion solutions of a typical flat sandwich composite panel are studied here. The propagative and evanescent solutions of the theory (30) presented in the third section are analyzed. Its 12 complex conjugated solutions correspond to propagative and evanescent waves in two opposite directions. Among these solutions, only eight are physically meaningful. The four absolute values of these solutions are represented in Figure 7. Additionally, three propagating waves asymptotes (computed using Eqs. (9) and (10)) are represented. It is observed that the main dispersion solution \((-**\text{-}\)-) has three asymptotical behaviors corresponding to panel’s bending, core’s shearing and the separate bending of skins. A second solution \((-\text{●-}\)-), always evanescent, has constant values at low and middle frequencies; its imaginary part tends asymptotically to the real part of the main solution at very high frequencies. At low and middle frequencies the third solution \((-\text{■-}\)-) is evanescent and its imaginary part is asymptotically equal to the real part of the main solution. At very high frequencies, a transition frequency in the dispersion solutions set can be observed; at this frequency the fourth \((-\text{-●-}\)-) and the third solutions became propagative.

The solutions represented in Figure 7 are reported in Figure 8 and compared to the dispersion relation’s solutions of a symmetrical laminate panel modeling type \(-\text{●-}\)- developed in the section 2. It is observed in Figure 8 that symmetrical laminate theory has just six true solutions (compared to eight for sandwich type \(-\text{■-}\)- modeling). The main solution has just two asymptotical behaviors: pure bending of the panel at low frequencies and core’s shearing at middle and high frequencies. This modeling type is not able to capture the separate pure bending of the skins at very high frequencies. The second dispersion
solution corresponding to the symmetrical laminate modeling type is not represented in Figure 8; it tends to values approaching infinity. The values of the third and fourth solutions as well as the transition frequencies are identical for both modeling types.

![Structural dispersion curves - Flat sandwich panel](image)

**Figure 7.** Propagative and evanescent wave solutions and asymptotes of a sandwich composite panel.

### 3.2. Group velocity and modal density

The behaviors observed on the dispersion solutions are analyzed in the context of the group velocity and the modal density calculation for a sandwich composite panel. Using the main solution of the dispersion relations (30) and (11) in the expressions presented in section 2.4, the group velocity and the modal density are computed. The group velocity results for both sandwich composite and symmetrical laminate modeling types are represented in Figure 9. Additionally, the asymptotical behaviors (15) of pure bending of the panel, core’s shearing and pure bending of the skins are represented. It is observed in Figure 9 that symmetrical laminate composite modeling gives similar results at low and middle frequencies; at very high frequencies the symmetrical laminate composite modeling is not able to capture separate pure bending of the skins’ behaviors and consequently the group velocity is underestimated.

The modal density behaviors are represented in Figure 10. The asymptotes are computed using the relations (15) and the definitions in section 2.4. It is observed that at very high frequencies the symmetrical laminate composite modeling overestimates the modal density of a typical sandwich composite panel.
Figure 8. Sandwich panel's dispersion curves: (—) symmetrical laminate approach and (—) sandwich-type modeling.

Figure 9. Group velocity and its asymptotical tendencies for a sandwich composite panel.
4. VALIDATION RESULTS

Comparisons of the sandwich composite panel theory with existing models are presented in this section. As a first example, the natural frequencies of a simply supported sandwich panel are computed with the theory presented in the third section and compared with finite elements results. The properties of the materials used for this example are presented in Table 3. The panel's side dimensions are 0.35×0.22m². The core's thickness is 0.0001m while the skins' thicknesses are 0.00045m. The panel is modeled in Nastran as an arrangement of a solid elements core bounded by shell elements for the skins with offsets. These results are plotted in Figure 11. Excellent agreement is observed between the results obtained with the present sandwich theory and Nastran. Note that the use of a composite element (CQUAD4) will lead to results similar to the presented laminate model and thus will overestimate the modal density at high frequencies.

Finally, the dispersion relation's main solution (30) is used within a Transfer Matrix Method (TMM) context to compute the total transmission loss of flat sandwich composite panel. A geometrical windowing method of the radiating field is used. This correction method is detailed in reference [9] and examples of its validation are given in references [9] and [10]. This method is compared here to the model proposed by Leppington et al. [15]. The theory presented in reference [15] was coded and the result is plotted in Figure 12. Excellent agreement is observed over the entire frequency range. Details of the experimental procedure and the panel properties are given in Ref. [15]. Note that the physical properties of the composite panel studied in Figure 12 are derived from the following parameters: $D_{11}=21.34$(Nm); $D_{12}+2D_{66}=27.78$(Nm); $D_{22}=330.84$(Nm); $m_s=4.87$(kg/m²); $L_x=0.9$m; $L_y=1.4$m; $\eta=0.01$. 

Figure 10. Modal density and its asymptotical tendencies for a sandwich composite panel.
Simply supported sandwich plate

Figure 11. Natural frequencies validation for a sandwich panel.

Transmission Loss - Flat composite panel

Figure 12. Transmission Loss validation for a sandwich composite panel.
Table 3. The physical properties of the materials used for the sandwich panel.

<table>
<thead>
<tr>
<th></th>
<th>Skins</th>
<th>Core</th>
</tr>
</thead>
<tbody>
<tr>
<td>E (Pa)</td>
<td>1.8 · 10^{11}</td>
<td>6.65 · 10^{3}</td>
</tr>
<tr>
<td>G (Pa)</td>
<td>6.767 · 10^{10}</td>
<td>2.5 · 10^{6}</td>
</tr>
<tr>
<td>ν</td>
<td>0.33</td>
<td>0.33</td>
</tr>
<tr>
<td>ρ (kg/m^3)</td>
<td>7720</td>
<td>2000</td>
</tr>
</tbody>
</table>

5. CONCLUSIONS AND DISCUSSIONS

A fast and accurate theory to model the vibro-acoustic behaviors of sandwich composite panels has been presented. The physical behavior of the panel is represented using a discrete lamina description. The model is developed in the context of a wave approach. It is shown that the dispersion curves are accurately estimated. Two modeling approaches were compared: symmetrical laminate composite and sandwich composite. It was observed that symmetrical laminate theory has an incomplete set of solutions: just six true solutions compared to eight for sandwich type modeling. Moreover, the main solution of symmetrical laminate theory was observed to have just two asymptotical behaviors. This modeling type was shown not to capture the separate pure bending of the skins at very high frequencies. Using the dispersion relation's solutions, the group velocity, the modal density, the natural frequencies as well as the total transmission loss were calculated. The acoustic transmission problem was represented within Finite Transfer Matrix Method (FTMM) context and successfully compared to experiments and to an asymptotical approach applied to flat composite panels.

Appendix: Main equations of the symmetrical laminate panel model

Using the notations represented in Figure 1, the integration limits used in the relations (A1) are computed as follows:

\[ M_x = \int_{-h/2}^{h/2} \sigma_x dz = \sum_{k=1}^{N} k \int_{h_k}^{h} \sigma_x z dz \]
\[ Q_x = \int_{-h/2}^{h/2} \tau_{xz} dz = \sum_{k=1}^{N} k \int_{h_k}^{h} \tau_{xz} z dz \]
\[ M_y = \int_{-h/2}^{h/2} \sigma_y dz = \sum_{k=1}^{N} k \int_{h_k}^{h} \sigma_y z dz \]
\[ Q_y = \int_{-h/2}^{h/2} \tau_{yz} dz = \sum_{k=1}^{N} k \int_{h_k}^{h} \tau_{yz} z dz \]

\[ M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} dz = \sum_{k=1}^{N} k \int_{h_k}^{h} \tau_{xy} z dz \]

Using the notations represented in Figure 1, the integration limits used in the relations (A1) are computed as follows:

\[ h_{hk} = \frac{h}{2} - \sum_{j=0}^{k-1} h_j; \quad h_k = \frac{h}{2} - \sum_{j=1}^{k} h_j; \]

where \( h \) is the total thickness of the panel, \( h_j \) the thickness of the lamina \( j \) and for \( j=0, h_j=h_0=0 \).

The mass per unit area and the rotational inertia are defined as follows:
\[ m_k = \sum_{k=1}^{N} \rho_k (h_{ak} - h_{bk}); \quad I_k = \sum_{k=1}^{N} \rho_k \frac{(h_{ak}^3 - h_{bk}^3)}{3}; \]  

(A3)

where, \( N \) is the total number of laminas and \( \rho_k \) is the mass density of lamina \( k \).

The general elastic constants of the symmetrical laminate composite panel \( Q_{ij}^k \), \( C_{ij}^k \) are defined as follows (Berthelot [7]):

\[
\begin{align*}
Q_{11}^k &= C_L \cos^4 \theta_k + C_T \sin^4 \theta_k + 2(C_{LT} + 2G_{LT}) \sin^2 \theta_k \cos^2 \theta_k \\
Q_{12}^k &= (C_L + C_T - 4G_{LT}) \sin^2 \theta_k \cos^2 \theta_k + C_{LT} (\cos^4 \theta_k + \sin^4 \theta_k) \\
Q_{16}^k &= (C_L - C_{LT} - 2G_{LT}) \sin \theta_k \cos \theta_k \sin \theta_k \cos \theta_k + (C_{LT} - C_T + 2G_{LT}) \sin^2 \theta_k \cos \theta_k \\
Q_{22}^k &= C_L \sin^4 \theta_k + C_T \cos^4 \theta_k + 2(C_{LT} + 2G_{LT}) \sin^2 \theta_k \cos^2 \theta_k \\
Q_{26}^k &= (C_L - C_{LT} - 2G_{LT}) \sin^3 \theta_k \cos \theta_k \sin \theta_k \cos \theta_k + (C_{LT} - C_T + 2G_{LT}) \sin \theta_k \cos^3 \theta_k \\
Q_{66}^k &= (C_L + C_T - 2(C_{LT} + G_{LT})) \sin^2 \theta_k \cos^2 \theta_k + G_{LT} (\cos^4 \theta_k + \sin^4 \theta_k)
\end{align*}
\]

(A4)

with, \( C_L^k = \left( \frac{E_L}{1 - \nu_{LT}^2}\right) \) \( C_T^k = \left( \frac{E_T}{1 - \nu_{LT}^2}\right) \) \( C_{LT}^k = \left( \frac{\nu_{LT} E_T}{1 - \nu_{LT}^2}\right) \);

and the elastic constants of transversal shearing of the core defined as follows (Berthelot [7]):

\[
\begin{align*}
C_{41}^k &= G_{TX}^k \cos^2 \theta_k + G_{LS}^k \sin^2 \theta_k \\
C_{51}^k &= (G_{TX}^k - G_{LS}^k) \sin \theta_k \cos \theta_k \\
C_{61}^k &= G_{LS}^k \cos^2 \theta_k + G_{TX}^k \sin^2 \theta_k
\end{align*}
\]

(A5)

REFERENCES