ON UNIQUENESS OF FILIPPOV'S SOLUTIONS FOR NON-SMOOTH SYSTEMS HAVING MULTIPLE DISCONTINUITY SURFACES WITH APPLICATIONS TO ROBOTIC CONTROL SYSTEMS

Q. Wu and H. Zeng
Department of Mechanical and Manufacturing Engineering
The University of Manitoba
Winnipeg, Manitoba, Canada R3T 5V6
Contact: cwu@cc.umanitoba.ca

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ABSTRACT

Analysis of the uniqueness of Filippov's solutions to non-smooth robotic control systems is important before the solutions can be sought. Such an analysis is extremely challenging when the discontinuity surface is the intersection of multiple discontinuity surfaces. The key step is to study the intersections of the convex sets from Filippov's inclusions and their associated sets containing vectors tangent to the discontinuity surfaces. For practical non-smooth robotic systems, due to their complexities, the determination of the intersections of these sets symbolically is extremely difficult if not impossible. In this paper, we propose a method such that the determinations of the intersections become feasible. Two examples of practical non-smooth robotic control systems are presented to demonstrate the efficacy of the method. The work contributes significantly to the analysis of non-smooth systems where the proof of the uniqueness of Filippov's solution is crucial to keep the mathematical model relevant to physical systems and to ensure the numerical solutions can be sought.

SPÉCIFICITÉ DE SOLUTIONS FILIPPOV POUR DES SYSTÈMES NON LISSES AYANT DE MULTIPLES SURFACES DE DISCONTINUITÉ AVEC DES APPLICATIONS AUX SYSTÈMES DE CONTRÔLE ROBOTIQUE

RÉSUMÉ

L'analyse de la spécificité de solutions Filippov aux systèmes de contrôle robotique non lisses est importante avant d'en arriver aux solutions. Une telle analyse présente d'importants défis lorsque la surface de discontinuité est l'intersection de multiples surfaces de discontinuité. L'étape principale consiste à étudier les intersections d'ensembles convexes des inclusions de Filippov et leurs ensembles associés contenant des vecteurs qui forment une tangente avec les surfaces de discontinuité. En raison de la complexité des systèmes pratiques robotiques non lisses, il est difficile – sinon impossible – de déterminer symboliquement les intersections de ces ensembles. Dans la présente, nous proposons une méthode permettant de déterminer les intersections. Deux exemples de systèmes de contrôle robotique non lisse sont présentés pour démontrer l'efficacité de la méthode. Le travail contribue de façon importante à l'analyse de systèmes non lisses lorsque la preuve de spécificité de la solution Filippov est essentielle pour veiller à la pertinence du modèle mathématique et pour s'assurer de trouver des solutions numériques.

1. INTRODUCTION

Non-smooth dynamic systems, described by ordinary differential equations with discontinuous terms, appear naturally and frequently in the robotic control field as well as many engineering applications. Typical examples include systems with stick-slip friction, robotic systems performing contact tasks, variable structure and optimal control systems. One fundamental issue, involved in the non-smooth systems, is that classical solution theories to ordinary differential equations are no longer valid because they require vector fields to be at least Lipschitz continuous. Non-smooth systems fail this requirement. Furthermore, one cannot even define a solution in the context of classical solution theories, much less discuss its existence, uniqueness and solve for the solution.

Filippov's solution theory [1-3] is one of the earliest and most conceptually straightforward approaches developed for the solution analysis of non-smooth systems, and has been applied by many researchers [4-8, 10, 11 and the references cited in]. In Filippov's work, a new definition of solutions was given, which is referred to here as Filippov's solution. Theorems were proven for existence, uniqueness and continuous dependence of Filippov's solutions on the initial conditions. Furthermore, conditions were proposed under which the numerical results converge to the Filippov's solution [3]. Note that many non-smooth systems possess more than one solution [9]. The proof of the uniqueness of Filippov's solutions is crucial for practical non-smooth robotic systems due to the following two reasons. Firstly, non-smooth systems are simplified mathematical models of robotic systems. To keep the models relevant to the physical systems, unique solution must be guaranteed [12]. Otherwise, questions such as under what conditions, which solution will show up, and which solution is relevant to the physical system, must be answered before the non-smooth mathematical models can be used to investigate the physical systems. Secondly, numerical simulations are an important part for investigating robotic systems. Based on the theorem developed by Filippov [3], the uniqueness of the solution is one of the key conditions for the convergence of numerical results to the solution.

To prove the uniqueness of Filippov's solutions, two scenarios arise. One is that the discontinuity surface is a single surface. Another one is that the discontinuity surface is the intersection of multiple discontinuity surfaces. The analysis of the uniqueness of Filippov's solutions for the first scenario is straightforward [1]. However, the analysis for the second scenario is extremely challenging due to the requirement of determining the intersections of the convex sets from Filippov's inclusions and their associated sets containing vectors tangent to the discontinuity surfaces [1,2]. To the best of our knowledge, the available methods for evaluating intersections of (convex) sets are all numerical methods. Since numerical solutions are not available before the proof of the uniqueness of Filippov's solutions [3], the numerical values of the elements of the above mentioned sets cannot be obtained. Furthermore, for robotic control systems, due to their complexities, evaluations of the intersections of the above mentioned sets symbolically are extremely difficult. On the other hand, many non-smooth robotic control systems possess more than one discontinuity surface, and investigation into the uniqueness of Filippov's solutions with intersecting discontinuity surfaces is inevitable. Therefore, although Filippov's solution theory has been developed, due to the lack of the tools in the analysis of the uniqueness of solutions, the applications of Filippov's solution theory to practical non-smooth robotic control systems has been severely restricted.

In this paper, we intend to fill the gap by developing a method to facilitate the use of Filippov's solution theory for the analysis of unique Filippov's solutions for non-smooth systems with intersecting discontinuity surfaces. Two examples of practical non-smooth robotic control systems are presented to demonstrate the efficacy of the method. In spite of the limitations of the method, as discussed later, the work contributes significantly to the studies of robotic systems using the computer modeling approach since many of the robotic systems are modeled as non-smooth systems.

2. MATHEMATICAL REVIEW

In this section, we review Filippov's solution concept and the theorem of the uniqueness of Filippov's solution when the discontinuity surface is an intersection of multiple discontinuity surfaces.

2.1. Filippov's Solution Concept [1,2]

In a bounded region G of $(t,\mathbf{x})$-space, where $\mathbf{x}=(x_1,...x_n)$, we consider the system:

$$\dot{\mathbf{x}} = f(\mathbf{x},t) \tag{1}$$

A. The region G is formed of a finite number of regions G_j ($j=1,2,...,r$), and the set M of points on the boundaries of the region G_j. In each G_j, the vector-valued function $f(\mathbf{x},t)$ is continuous up to the boundary, *i.e.*, there is a function $f^j(\mathbf{x},t)$ defined and continuous in G_j and on its boundary and equal to $f(\mathbf{x},t)$ in G_j.

B. The boundary of each region G_j is piecewise smooth; *i.e.*, it is formed of a finite number of parts of smooth n-dimensional surfaces bounded by a finite number of smooth surfaces of lower dimension and points.

C. For all values of t except those in a finite set T of measure zero, each point $(\mathbf{x},t)$ of the boundary of each G_j is also on the boundary of the section of this region by the plane $t = const.$

A solution of Eq. (1) is understood to be an absolutely continuous vector-valued function $\mathbf{x}(t)$ defined on an interval I and satisfying the relation

$$\dot{\mathbf{x}} \in F(\mathbf{x}(t),t) \tag{2}$$

almost everywhere on I; here $F(\mathbf{x},t)$ is the smallest closed convex set containing all limit values of the vector-valued function $f(\mathbf{x}^*,t)$ when $(\mathbf{x}^*,t)$ tends to $(\mathbf{x},t)$ through points of G_j in the plane $t = const.$

2.2. Uniqueness of Filippov's Solution [2]

To facilitate the presentation of the Filippov's solution concept, some terminology [1,2] is reviewed first. Let a domain $G \subset R^n$ be separated by smooth discontinuity surfaces S_i^m ($i=1,2,...I$) ($0 \le m < n$) into S_j^n ($j=1,2,...,J$). The superscript denotes the dimension and the subscript denotes the surface. S_i^n represents a continuous region of $f(\mathbf{x},t)$. S_i^ℓ is the intersecting discontinuity surface under study, S_i^p $(p=\ell+1,...,n)$ are surfaces abutting S_i^ℓ. The edge of each surface does not belong to the surface and the edge consists of a finite number of smooth surfaces of smaller dimension and points.

For $\mathbf{x}\in S_i^\ell$, $F_i^\ell(\mathbf{x},t)$ is defined by (2). $P_i^\ell(\mathbf{x})$ is the set of all vectors parallel to the ℓ-dimensional tangent plant to S_i^ℓ at $\mathbf{x}$ including the null vector. Based on [2], a vector $\overrightarrow{AB}$ is tangent to surface at the point A, if

$$\overrightarrow{AB}=\left|\overrightarrow{AB}\right| lim(\frac{\overrightarrow{AC_k}}{\left|\overrightarrow{AC_k}\right|}) \text{ as } C_k\in S_i^\ell, C_k\to A \text{ as } k\to\infty \tag{3}$$

For each $\mathbf{x}\in S_i^\ell$, set $K_i^\ell(\mathbf{x},t)$ is defined as

$$K_i^\ell(\mathbf{x},t)=F_i^\ell(\mathbf{x},t)\cap P_i^\ell(\mathbf{x}) \tag{4}$$

According to [2], if $\mathbf{x}$ is on the edge (boundary) of S_i^ℓ, $F_i^\ell(\mathbf{x},t)$ is the limit position of the manifold $F_i^\ell(\mathbf{x}',t)$ when $\mathbf{x}'\in S_i^\ell$, $\mathbf{x}'\to\mathbf{x}$; $H_i^\ell(\mathbf{x},t)$ is the set of vectors of $K_i^\ell(\mathbf{x},t)$ at boundary point $\mathbf{x}$. The above definitions are valid for $m=p$ $(p=\ell+1,......,n)$, *i.e.*, on the surfaces abutting the intersecting discontinuity surfaces, S_i^ℓ.

Theorem of unique Filippov's solution [2]: In a bounded region G of (**x**, t)-space, suppose that, for $t_1\le t\le t_2$: (*i*) each solution of Eq. (1) goes from one set S_i^ℓ into another only a finite number of times; (*ii*) there is right-sided uniqueness up to the boundary in each S_i^ℓ; (*iii*) each S_i^ℓ possesses one of the following two properties: (a) for all S_j^p abutting S_i^ℓ the sets $H_j^p(\mathbf{x},t)$ are empty for all $\mathbf{x}\in S_i^\ell$; (b) only one of them is non-empty and $K_i^\ell(\mathbf{x},t)$ is empty. Then, Eq. (1) has the right-sided uniqueness in G for $t_1\le t\le t_2$.

Remarks:

1. In case (a), a solution reaching S_i^ℓ remains there, and set $K_i^\ell(\mathbf{x},t)$ is nonempty, while in case (b) such a solution immediately moves from S_i^ℓ to an abutting S_j^p for which $H_j^p(\mathbf{x},t)$ is nonempty. Thus the determination of the surface with a non-empty set is crucial for seeking the numerical results.
2. Condition (*iii*) appears different from the one in [1], but they are equivalent. Both indicate that among all sets, $K_i^\ell(\mathbf{x},t)$ and $H_i^p(\mathbf{x},t)$, one and only one of them is non-empty, the rest of them are empty. In this work, we use the conditions in [2] since it is more constructive and explicit.

3. METHODOLOGY

Based on the above theorem, the emptiness of sets $K_i^\ell(\mathbf{x},t)$ and $H_j^p(\mathbf{x},t)$ must be examined to study the uniqueness of Filippov's solution. This is challenging because these sets are functions of the states and time, and their numerical values are not available since the numerical solution can not be obtained before the uniqueness of the solution is guaranteed [3]. To remedy the problem, we first transform the system to a new state space where the discontinuity surfaces can be written in a special form. We then expand the sets associated with Filippov's inclusion to $\hat{F}_i^m(\mathbf{x}',t)$ ($m=\ell$ or p), such that the determinations of the intersections become feasible. Finally, a theorem is stated for the uniqueness of Filippov's solution in terms of the expanded sets.

3.1. State Space Transformation

In the course of this work, we noticed that if the discontinuity surface can be described as $x_i = 0$ $(i=1,2,...,r)$, each coordinate of the vectors in set $P_i^m(\mathbf{x})$ is either positive, negative, zero or arbitrary. For example, considering a non-smooth system in a plane where each coordinate axis is a single discontinuity surface and the origin is the intersecting discontinuity surface, each set $P_i^m(\mathbf{x})$ $(m=0,1,2)$ can be one of the followings: the origin, positive or negative coordinate axis, and one of the four quadrants. The special form of sets $P_i^m(\mathbf{x})$ makes the symbolic evaluation of the intersections of sets $F_i^m(\mathbf{x},t)$ and $P_i^m(\mathbf{x})$ possible. Thus, we propose to transform the original state space $(\mathbf{x},t)$ to a new state space $(\mathbf{s},t)$ such that each discontinuity surface can be described as $s_i = 0$ $(i=1,2,...,r)$. Sets $F_i^m(\mathbf{x},t)$ are to be determined in the new state space $(\mathbf{s},t)$. Note that the elements of sets $F_i^m(\mathbf{s},t)$ are vectors, and the distinct co-ordinates of the vectors are named variables in this paper. It is often that several co-ordinates are the same variable.

3.2. Expansion of sets $F_i^m(\mathbf{s},t)$

In this work, set $F_i^m(\mathbf{s},t)$ is expanded to $\hat{F}_i^m(\mathbf{s},t)$ such that (1) the coordinates of each vertex of $\hat{F}_i^m(\mathbf{s},t)$ are the maximum or minimum coordinates of those from the original set $F_i^m(\mathbf{s},t)$, and (2) the surfaces and the edges of $\hat{F}_i^m(\mathbf{s},t)$ are parallel to their corresponding coordinate surfaces and axes, respectively. The expansion is illustrated using a simple convex set in a plane as shown in Figure 1(a). Consider a convex set, $F_1^2(\mathbf{s},t)$, which can be represented as a triangle ABC with A, B and C as the vertices. Firstly, $F_1^2(\mathbf{s},t)$ is expanded by generating a rectangle, $\hat{A}\hat{B}\hat{C}\hat{D}$, with each side parallel to the corresponding co-ordinate axis as shown in Figure 1(a). Secondly, the smallest rectangle $\hat{F}_1^2(\mathbf{s},t)$ is created, which contain all corresponding sets $F_1^2(\mathbf{s},t)$ as the state vector, $\mathbf{s}$, takes all possible values.

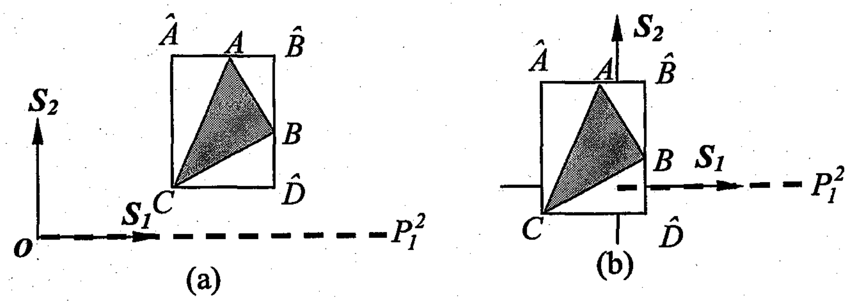

Figure 1. An illustrative example of the proposed method. (a) expansion of set $\hat{F}_1^2(\mathbf{s},t)$, and (b) intersection of sets $\hat{F}_1^2(\mathbf{s},t)$ and $P_1^2(\mathbf{s})$.

The algorithm to determine set $\hat{F}_1^2(\mathbf{s},t)$ consists of two parts. One is to determine the maximum and minimum co-ordinates of vertices for $F_1^2(\mathbf{s},t)$. Referring to Figure 1(a), we have

$$\begin{aligned} X_{min} &= min(X_A, X_B, X_C) & X_{max} &= max(X_A, X_B, X_C) \\ Y_{min} &= min(Y_A, Y_B, Y_C) & Y_{max} &= max(Y_A, Y_B, Y_C) \end{aligned} \tag{5}$$

where X and Y are the coordinates of the vertices. Another part is to create rectangles using the maximum and minimum coordinates as the vertices. For the above example, we have that

$$\hat{F}_i^2 = \overline{co}\left\{\begin{pmatrix} X_{min} \\ Y_{min} \end{pmatrix} \begin{pmatrix} X_{max} \\ Y_{max} \end{pmatrix} \begin{pmatrix} X_{max} \\ Y_{min} \end{pmatrix} \begin{pmatrix} X_{min} \\ Y_{max} \end{pmatrix}\right\} \tag{6}$$

Consequently, set $\hat{K}_i^\ell(\mathbf{s},t)$ is defined as:

$$\hat{K}_i^\ell(\mathbf{s},t) = \hat{F}_i^\ell(\mathbf{s},t) \cap P_i(\mathbf{x}) \tag{7}$$

and set $\hat{H}_i^p(\mathbf{s},t)$ is the set of vectors of $\hat{K}_i^p(\mathbf{s},t)$ at boundary point $\mathbf{s}$.

3.3. Theorem for the Uniqueness of Filippov's Solution

In this section, the theorem for the uniqueness of Filippov's solution for non-smooth systems with intersecting discontinuity surfaces is stated in terms of the expanded sets. The proof of the theorem is also given, and finally three remarks are discussed.

Theorem: Let the non-smooth system, shown in Eq. (1), be locally bounded and have at least one Filippov's solution. Let the discontinuity surface, S_i^m, be of the following special form:

$$S_i^m := \{\mathbf{x}, x_j = 0, j = 1, \ldots, J\} \tag{8}$$

where $m = \ell \; or \; p$. Suppose that, for $t_1 \le t \le t_2$: (*i*) each solution of Eq. (1) goes from one set S_i^ℓ into another only a finite number of times; (*ii*) there is right-sided uniqueness up to the boundary in each S_i^ℓ; (*iii*) each S_i^ℓ possesses one of the following two properties: (a) for all S_j^p abutting S_i^ℓ, the sets $\hat{H}_j^p(\mathbf{x},t)$ are empty for all $\mathbf{x} \in S_i^\ell$, and $\hat{K}_i^\ell(\mathbf{x},t)$ is non-empty; (b) only one of $\hat{H}_j^p(\mathbf{x},t)$ is non-empty and $\hat{K}_i^\ell(\mathbf{x},t)$ is empty. Then, Eq. (1) has the right-sided uniqueness in G for $t_1 \le t \le t_2$.

Proof: Since conditions (*i*) and (*ii*) are identical to those from Filippov's original Theorem [2], it is to prove that the satisfaction of conditions (a) and (b) in condition (*iii*) leads to the satisfaction of their counterparts in Filippov's original theorem [2] as follows:

Since $\hat{F}_i^m(\mathbf{x},t)$ is the expansion of set $F_i^m(\mathbf{x},t)$, as discussed in Section 3.2, we have that

$$F_i^m(\mathbf{x},t) \subseteq \hat{F}_i^m(\mathbf{x},t) \tag{9a}$$

$$K_i^m(\mathbf{x},t) \subseteq \hat{K}_i^m(\mathbf{x},t) \tag{9b}$$

$$H_i^m(\mathbf{x},t) \subseteq \hat{H}_i^m(\mathbf{x},t) \tag{9c}$$

Considering (a) of condition (*iii*), if sets $\hat{H}_i^p(\mathbf{x},t)$ are empty, their corresponding sets $H_i^p(\mathbf{x},t)$ are empty. Thus, it is to prove that if $\hat{K}_i^\ell(\mathbf{x},t)$ is not empty, its corresponding set $K_i^\ell(\mathbf{x},t)$ must be non-empty. This can be proven by contradiction. Assuming that set $K_i^\ell(\mathbf{x},t)$ is empty, all sets $H_i^p(\mathbf{x},t)$ and $K_i^\ell(\mathbf{x},t)$ are empty. Thus there is no solution exist, which conflict with the assumption that there is at least one solution. Thus, set $K_i^\ell(\mathbf{x},t)$ is non-empty, and (a) of condition (*iii*) in Filippov's original theorem [2] is satisfied.

Similarly, if set $\hat{K}_i^\ell(\mathbf{x},t)$ is empty, so is the set $K_i^\ell(\mathbf{x},t)$. Furthermore, it can be proven by contradiction following the same procedure that if only one of the sets $H_i^p(\mathbf{x},t)$, namely $\hat{H}_i^{p*}(\mathbf{x},t)$, is not empty, its corresponding set $H_i^{p*}(\mathbf{x},t)$ must be non-empty. Thus (b) of conditions (*iii*) of Filippov's theorem [2] is satisfied.

The above proof shows that the satisfaction of the conditions of the proposed theorem guarantees the satisfaction of all the conditions of Filippov's original theorem [2]. Thus, Eq. (1) has the right-sided uniqueness in G for $t_1 \le t \le t_2$.

Remarks:

1. The evaluation of the emptiness of sets $\hat{H}_i^p(\mathbf{x},t)$ and $\hat{K}_i^\ell(\mathbf{x},t)$ is based on the fact that sets $\hat{F}_i^m(\mathbf{x},t)$ and $P_i^m(\mathbf{s})$ are manifolds with each edge parallel to the corresponding coordinate axis. The emptiness of the intersection of the expanded sets $\hat{H}_i^p(\mathbf{x},t)$ and $\hat{K}_i^\ell(\mathbf{x},t)$ can then

be determined by evaluating the overlapping of the corresponding edges. If one pair of the corresponding edges does not overlap, the intersection of the expanded sets is empty. Consequently, the intersection of the original sets is also empty.

2. Since all elements of sets $F_i^m(\mathbf{x},t)$ are symbolic, the state space is divided into regions. Due to the fact that each coordinate of the vectors in set $P_i^m(\mathbf{x})$ is either positive, negative, zero or arbitrary, the state space is partitioned such that each variable, defined as the distinct co-ordinate of the vectors in set $\hat{F}_i^m(\mathbf{x},t)$, will take all possible signs, and the intersection of sets $\hat{F}_i^m(\mathbf{x},t)$ and $P_i^m(\mathbf{x})$ is evaluated in each region. Due to the high dimension of the state space and the large number of regions where the intersections of sets $\hat{F}_i^m(\mathbf{x},t)$ and $P_i^m(\mathbf{x})$ ($m = \ell$ *and* p) are evaluated, a computer program is developed.
3. The proposed theorem is more restrictive than Filippov's original theorem. It can be seen that if sets $\hat{H}_i^p(\mathbf{x},t)$ and $\hat{K}_i^\ell(\mathbf{x},t)$ are empty, sets $H_i^p(\mathbf{x},t)$ and $K_i^\ell(\mathbf{x},t)$ are empty. However, if set $\hat{H}_i^p(\mathbf{x},t)$ and $\hat{K}_i^\ell(\mathbf{x},t)$ are not empty, sets $H_i^p(\mathbf{x},t)$ and $K_i^\ell(\mathbf{x},t)$ may still be empty, as shown in Figure 1(b), and no conclusion can be drawn. For these inconclusive cases, recommendations are made in the next section. In spite of the more restrictive outcome, our theorem is appealing since the numerical forms of sets $F_i^m(\mathbf{x},t)$ and $P_i^m(\mathbf{x})$ are not available and no general method exists to symbolically evaluate their intersections.

4. CASE STUDIES

To demonstrate the efficacy of the proposed theorem, two examples of practical non-smooth robotic systems, taken from literature, are presented.

4.1. Lyapunov Stability Control of a One-Link Base-Excited Robot Arm [7]

A robust Lyapunov feedback control has been developed [7] to stabilize a base-excited one-link robot arm about the upright position. The arm has two degrees of rotational freedom, and the base can move freely in the three dimensional space. The physical model is shown in Figure 2, and the dynamic equations are shown below:

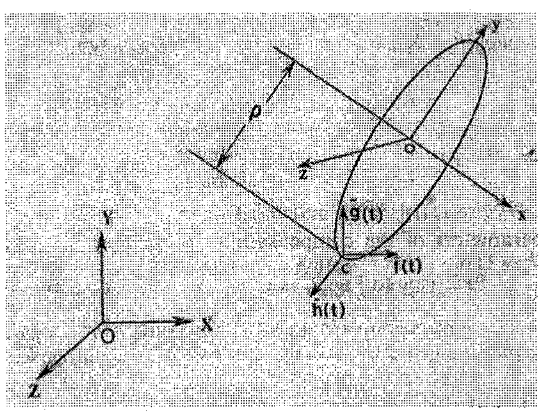

Figure 2. Inverted pendulum model

$$\ddot{\theta} = \frac{1}{(1-\mu \sin^2 \psi)}(\mu \sin(2\psi)\dot{\theta}\dot{\psi} + c_0(g + \ddot{g}(t))\sin\theta\cos\psi - c_0\ddot{f}(t)\cos\psi\cos\theta + \frac{1}{A}M_\theta)$$
$$\ddot{\psi} = -\frac{1}{2}\mu \sin(2\psi)\dot{\theta}^2 + c_0(g + \ddot{g}(t))\sin\psi\cos\theta - c_0\ddot{h}(t)\cos\psi + c_0\ddot{f}(t)\sin\psi\sin\theta + \frac{1}{A}M_\psi \tag{10}$$

where θ and ψ are Euler angles, defined in [7]. ρ is the distance between the gravity center and the base point. I_x, I_y and I_z are the mass moments of inertia and $I_x = I_z = I_0 > I_y$. $A = I_0 + m\rho^2$, $\mu = 1 - I_y / A$ and $c_0 = m\rho / A$. g is the gravitational acceleration ($9.8m/s^2$). $\ddot{f}(t)$, $\ddot{g}(t)$ and $\ddot{h}(t)$ are three acceleration components of the base point in X, Y and Z directions, as shown in Figure 2. M_θ and M_ψ are the control torques applied at the base point.

Due to limited available Lyapunov functions, a discontinuous Lyapunov feedback controller was designed and is shown as follows:

$$M_\theta = -(k_\theta + m\rho g)\theta - K_{d\theta}\dot{\theta} - m\rho G_0\, sgn(\dot{\theta})|\theta| - m\rho F_0\, sgn(\dot{\theta}))$$
$$M_\psi = -(k_\psi + m\rho g)\psi - K_{d\psi}\dot{\psi} - m\rho(G_0 + F_0)sgn(\dot{\psi})|\psi| - m\rho H_0 \mathrm{sgn}(\dot{\psi}) \tag{11a}$$

where $k_\theta, k_\psi, K_{d\theta}$ and $K_{d\psi}$ are constant control gains. F_0 , G_0 and H_0 are the upper bounds of the base point accelerations, $|\ddot{f}(t)|$, $|\ddot{g}(t)|$ and $|\ddot{h}(t)|$. sgn$(\dot{\theta})$ and sgn$(\dot{\psi})$ are discontinuous functions defined as

$$sgn(\gamma) = \begin{cases} 1 & \text{when } \gamma > 0 \\ -1 & \text{when } \gamma \le 0 \end{cases} \tag{11b}$$

which causes the dynamic system to be non-smooth. Assuming that $x_1 = \theta$ and $x_2 = \psi$, the state space model is shown below:

$$\dot{x}_1 = x_3$$
$$\dot{x}_2 = x_4$$
$$\dot{x}_3 = \frac{1}{1-\mu \sin^2 x_2}(\mu x_3 x_4 \sin(2x_2) + c_0(g + \ddot{g}(t))\sin x_1 \cos x_2 - (\frac{k_\theta}{A} + c_0 g)x_1 - \frac{K_{d\theta}}{A}x_3$$
$$- c_0 G_0\, sgn(x_3)|x_1| - c_0\ddot{f}(t)\cos x_1 \cos x_2 - c_0 F_0\, sgn(x_3) - \frac{K_1}{A}\tanh(\alpha_1 x_1)) \tag{12}$$
$$\dot{x}_4 = -\frac{1}{2}\mu x_3^2 \sin(2x_2) + c_0(g + \ddot{g}(t))\sin x_2 \cos x_1 - (\frac{k_\psi}{A} + c_0 g)x_2 - c_0\ddot{h}(t)\cos x_2 - c_0 H_0\, sgn(x_4)$$
$$+ c_0\ddot{f}(t)\sin x_1 \sin x_2 - c_0(G_0 + F_0)sgn(x_4)|x_2| - \frac{K_{d\psi}}{A}x_4 - \frac{K_2}{A}\tanh(\alpha_2 x_2)$$

There are three discontinuity surfaces:

$$S_1^3 := \{x; x_3 = 0 \,\&\, x_4 \neq 0\} \quad S_2^3 := \{x; x_3 \neq 0 \,\&\, x_4 = 0\} \quad S_1^2 := \{x; x_3 = 0 \,\&\, x_4 = 0\} \qquad (13)$$

The discontinuity surface, S_1^2, is the intersection of two surfaces S_1^3 and S_2^3. We will present the detailed procedure for proving the uniqueness of Filippov's solution on the intersecting discontinuity surface, S_1^2. Note that the equations of discontinuity surfaces, shown in (13), are in the desired form. Thus, transformation of state space is not required. The solution region G is divided into four regions:

$$\begin{array}{ll} S_1^4 := \{x; x_3 > 0 \,\&\, x_4 > 0\} & S_2^4 := \{x; x_3 > 0 \,\&\, x_4 < 0\} \\ S_3^4 := \{x; x_3 < 0 \,\&\, x_4 > 0\} & S_4^4 := \{x; x_3 < 0 \,\&\, x_4 < 0\} \end{array} \qquad (14)$$

The above regions are bounded by smooth surfaces denoted by S_i^m ($i=1,2,3,4$ and $m=2,3$):

$$\begin{array}{lll} S_1^3 := \{\mathbf{x}; x_3 = 0 \,\&\, x_4 > 0\} & S_2^3 := \{\mathbf{x}; x_3 = 0 \,\&\, x_4 < 0\} & S_3^3 := \{\mathbf{x}; x_3 > 0 \,\&\, x_4 = 0\} \\ S_4^3 := \{\mathbf{x}; x_3 < 0 \,\&\, x_4 = 0\} & S_1^2 := \{\mathbf{x}; x_3 = 0 \,\&\, x_4 = 0\} & \end{array} \qquad (15)$$

In addition, sets $P_i^m(\mathbf{x})$ has the following form:

$$P_1^3 = \left|\overrightarrow{AB}\right| \overline{co} \begin{Bmatrix} \alpha \\ \beta \\ 0 \\ \gamma \end{Bmatrix} \quad P_2^3 = \left|\overrightarrow{AB}\right| \overline{co} \begin{Bmatrix} \alpha \\ \beta \\ 0 \\ \lambda \end{Bmatrix} \quad P_3^3 = \left|\overrightarrow{AB}\right| \overline{co} \begin{Bmatrix} \alpha \\ \beta \\ \gamma \\ 0 \end{Bmatrix} \quad P_4^3 = \left|\overrightarrow{AB}\right| \overline{co} \begin{Bmatrix} \alpha \\ \beta \\ \lambda \\ 0 \end{Bmatrix} \quad P_1^4 = \left|\overrightarrow{AB}\right| \overline{co} \begin{Bmatrix} \alpha \\ \beta \\ \gamma \\ \gamma \end{Bmatrix}$$

$$P_2^4 = \left|\overrightarrow{AB}\right| \overline{co} \begin{Bmatrix} \alpha \\ \beta \\ \gamma \\ \lambda \end{Bmatrix} \quad P_3^4 = \left|\overrightarrow{AB}\right| \overline{co} \begin{Bmatrix} \alpha \\ \beta \\ \lambda \\ \gamma \end{Bmatrix} \quad P_4^4 = \left|\overrightarrow{AB}\right| \overline{co} \begin{Bmatrix} \alpha \\ \beta \\ \lambda \\ \lambda \end{Bmatrix} \quad P_1^2 = \left|\overrightarrow{AB}\right| \overline{co} \begin{Bmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{Bmatrix} \qquad (16a)$$

where $\left|\overrightarrow{AB}\right|$ is the length of the vector $\overrightarrow{AB}$. Variables α, β, γ and λ satisfy the conditions:

$$-1 \le \alpha \le 1 \quad -1 \le \beta \le 1 \quad 0 \le \gamma \le 1 \quad -1 \le \lambda \le 0 \qquad (16b)$$

Sets $P_i^m(\mathbf{x})$ ($m=3,4$) contain all vectors on the m-dimensional surface tangential to S_1^2. All $P_i^m(\mathbf{x})$ include the null vector. With the arbitrary positive number $\left|\overrightarrow{AB}\right|$, $\left|\overrightarrow{AB}\right|\alpha$ and $\left|\overrightarrow{AB}\right|\beta$ are arbitrary numbers between positive and negative infinities, and $\left|\overrightarrow{AB}\right|\gamma$ and $\left|\overrightarrow{AB}\right|\lambda$ are arbitrary positive and negative numbers, respectively. Based on Filippov's original theorem shown in

Section 2, the emptiness of sets $K_i^\ell(\mathbf{x},t)$ and $H_j^p(\mathbf{x},t)$ must be examined. Thus, sets $F_i^m(\mathbf{x},t)$ ($m=\ell$ and p) from Filippov's differential inclusion, defined in (2), are determined as shown below, where *B, C, D* and *E* have been defined in [7]:

$$F_1^3=\overline{co}\left(\left\{\begin{bmatrix}0\\0\\B+D\\C-E\end{bmatrix},\begin{bmatrix}0\\0\\B-D\\C-E\end{bmatrix}\right\}\right)\quad F_2^3=\overline{co}\left(\left\{\begin{bmatrix}0\\0\\B+D\\C+E\end{bmatrix},\begin{bmatrix}0\\0\\B-D\\C+E\end{bmatrix}\right\}\right)\quad F_3^3=\overline{co}\left(\left\{\begin{bmatrix}0\\0\\B-D\\C+E\end{bmatrix},\begin{bmatrix}0\\0\\B-D\\C-E\end{bmatrix}\right\}\right)$$

$$F_4^3=\overline{co}\left(\left\{\begin{bmatrix}0\\0\\B+D\\C+E\end{bmatrix},\begin{bmatrix}0\\0\\B+D\\C-E\end{bmatrix}\right\}\right)\quad F_1^4=\left(\left\{\begin{bmatrix}0\\0\\B-D\\C-E\end{bmatrix}\right\}\right)\quad F_2^4=\left(\left\{\begin{bmatrix}0\\0\\B-D\\C+E\end{bmatrix}\right\}\right)\quad F_3^4=\left(\left\{\begin{bmatrix}0\\0\\B+D\\C-E\end{bmatrix}\right\}\right)$$

$$F_4^4=\left(\left\{\begin{bmatrix}0\\0\\B+D\\C+E\end{bmatrix}\right\}\right)\quad F_1^2=\overline{co}\left(\left\{\begin{bmatrix}0\\0\\B-D\\C-E\end{bmatrix},\begin{bmatrix}0\\0\\B-D\\C+E\end{bmatrix},\begin{bmatrix}0\\0\\B+D\\C+E\end{bmatrix},\begin{bmatrix}0\\0\\B+D\\C-E\end{bmatrix}\right\}\right)\tag{17}$$

There are four variables, $(B+D)$, $(B-D)$, $(C+E)$, and $(C-E)$ in $F_i^m(\mathbf{x},t)$. Note that sets $F_i^m(\mathbf{x},t)$ are in the desired form that each edge is parallel to its corresponding co-ordinate axis.

Considering the sub-sets of $P_i^m(\mathbf{x})$ containing the vectors with the 1st and 2nd coordinates, which are between positive and negative infinities, their intersections with the corresponding sub-sets in $F_i^m(\mathbf{x},t)$ will never be empty. We then consider the sub-sets having the vectors with the 3rd and 4th coordinates from sets $F_i^m(\mathbf{x},t)$ and $P_i^m(\mathbf{x})$. The coordinates of the vectors in sub-sets $P_i^m(\mathbf{x})$ are either positive, zero or negative, thus, the state space is divided such that the corresponding coordinates from sub-set $F_i^m(\mathbf{x},t)$ can take all possible signs, and the intersection of $F_i^m(\mathbf{x},t)$ and $P_i^m(\mathbf{x})$ will be evaluated in each region. Thus, the total number of cases for evaluation is $3^4=81$. However, since $D>0$ and $E>0$ [7], the following constraints are imposed: $B+D>B-D$ and $C+E>C-E$. The above information is provided to the computer program. The summary of the output file is detailed in Table 1, which shows that due to the two constraints imposed on the variables of sets $F_i^m(\mathbf{x},t)$, 56 cases are impossible and are automatically removed. Among the remaining 25 possible cases, there is one and only one non-empty set. The emptiness of the expanded sets, $\hat{K}_i^\ell(\mathbf{x},t)$ or $\hat{H}_j^p(\mathbf{x},t)$, guarantees the emptiness of the original sets, $K_i^\ell(\mathbf{x},t)$ and $H_j^p(\mathbf{x},t)$. Thus, the Filippov's solution to Eq. (12), is unique.

The intersection of sets $F_i^m(\mathbf{x},t)$ and $P_i^m(\mathbf{x})$ has been determined manually in [7]. Identical results were achieved. This validates the method and the program developed here. In addition, our program identifies the non-empty set as the solution trajectory reaches the intersection of discontinuity surface. This information is crucial in finding numerical solutions as it indicates the surface that the solution trajectory will move to, and the possible value of the rate of the states.

Table 1 Summary of the results for Case study 1.

Information	Number of cases
Total cases	81
Impossible cases	56
One and only one non-empty set	25
More than one non-empty set	0

4.2. Contact task Control of Hydraulic Actuators [10]

A Lyapunov-based control law was designed for contact task control of an electro-hydraulic actuator [10]. The controller allows the hydraulic actuator to follow a free space trajectory and then to make and to maintain the contact with the environment for exerting a desired force. The schematic of the hydraulic actuator and the environment is shown in Figure 3 and the state space model is shown below [10]:

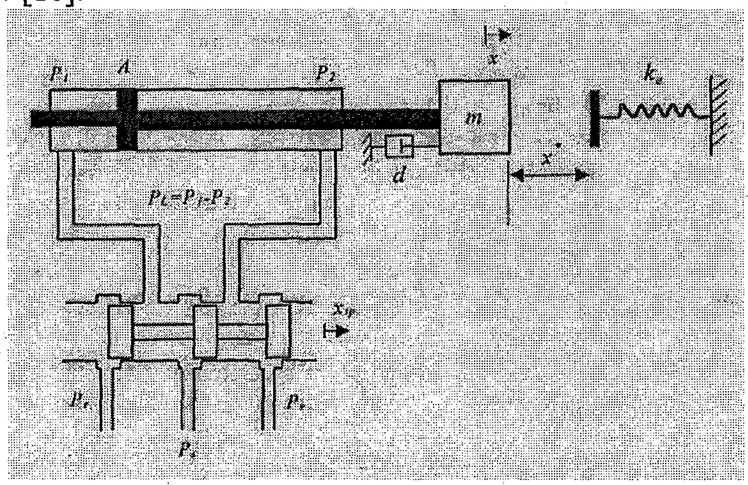

Figure 3. Schematic of hydraulic actuator and environment

$$
\begin{aligned}
\dot{x} &= v \\
\dot{v} &= \frac{A}{m}P_L - \frac{d}{m}v - \frac{f_e}{m} \\
\dot{P}_L &= \frac{1}{C}\left(-A\dot{x} + \frac{c_d}{\sqrt{\rho}} w x_{sp}\sqrt{P_s - sgn(x_{sp})P_L}\right) \\
\dot{x}_{sp} &= \frac{-1}{\tau}x_{sp} + \frac{k_{sp}}{\tau}u
\end{aligned}
\quad (18a)
$$

where m and x are the mass and the displacement of the moving part, respectively. d is the equivalent viscous damping coefficient, P_L is the load pressure, and A is the piston area. The contact force, f_e, is expressed as follows:

$$f_e = \begin{cases} 0 & x \le x^* \\ k_e(x - x^*) & x > x^* \end{cases} \tag{18b}$$

where k_e is the stiffness of the environment and x^* denotes the initial location of the piston head with respect to the environment. c_d is the orifice coefficient of discharge, w is the orifice area gradient, ρ is the hydraulic fluid density, P_s is the pump pressure. $C = V_t / 4\beta$ is the hydraulic compliance with V_t as the total compressed fluid volume and β as the effective bulk modulus of the system. x_{sp} is the spool displacement, u is the valve input voltage. k_{sp} and τ are the valve gain and time constant, respectively. The control laws for the non-contact and contact phases are designed separately. For the non-contact phase, assuming that

$$e_1 = x - x^d \qquad e_2 = \dot{x} \qquad e_3 = P_L \qquad e_4 = x_{sp} \tag{19a}$$

the controller is as follow:

$$u_P = -K_1[(1 + K_2)e_3 + K_2\overline{K}e_1]\sqrt{P_s - sgn(e_4)(e_3)} \tag{19b}$$

where K_1 and K_2 are positive constant gains and $\overline{K} = A / C$. $sgn(e_4)$ is a sign function as defined in Eq. (11b).

For the contact phase, assuming that

$$\tilde{e}_1 = x - \tilde{x}_d \qquad e_2 = \dot{x} \qquad \tilde{e}_3 = P_L - P^d \qquad e_4 = x_{sp} \tag{20a}$$

where P^d and $\tilde{x}_d$ are related to the desired reference force, f^d, as

$$p^d = \frac{f^d}{A}, \qquad \tilde{x}^d = x^* + \frac{f^d}{k_e} \tag{20b}$$

The controller was designed as:

$$u_f = -K_1\tilde{e}_3\sqrt{P_s - sgn(e_4)(\tilde{e}_3 + \frac{f_d}{A})} \tag{20c}$$

During the transition from the non-contact phase to the contact phase, the actuator is driven in free space toward a predefined reference position. Once an environment is encountered, the controller attempts to maintain contact and to exert a desired force. The state space model with respect to the state definitions (20a) is described as follows:

$$\begin{aligned}
\dot{\tilde{e}}_1 &= e_2 \\
\dot{e}_2 &= \frac{A}{m}\tilde{e}_3 - \frac{d}{m}e_2 - \frac{\tilde{f}_e}{m} \\
\dot{\tilde{e}}_3 &= -\frac{A}{C}e_2 + \frac{c_d w}{\sqrt{\rho C}} e_4 \sqrt{P_s - sgn(e_4)(\tilde{e}_3 + \frac{f^d}{A})} \\
\dot{e}_4 &= -\frac{1}{\tau}e_4 + \frac{k_{sp}}{\tau}u_t
\end{aligned} \tag{21a}$$

where

$$\tilde{f}_e = \begin{cases} -f^d & x \le x^* \\ \tilde{e}_1 k_e & x > x^* \end{cases} \tag{21b}$$

The controller, u_t, switches between u_P and u_f, i.e., $u_t = u_P$ as $f_e = 0$ and $u_t = u_f$ as $f_e > 0$, where u_P and u_f are shown in (19b) and (20c), respectively.

The above system, shown in (21), is non-smooth due to the switching of the controller when the actuator contacts the environment and the discontinuous control term, $\tilde{e}_3\sqrt{P_s - sgn(e_4)(\tilde{e}_3 + \frac{f_d}{A})}$. Due to the complexity of the non-smooth system, the uniqueness of Filippov's solution has not been properly studied.

There are three discontinuity surfaces for the transition phase, which are shown below:

$$\begin{aligned}
S_1 &:= \{ e : e_4 = 0 \text{ and } \tilde{e}_1 \ne x^* - \tilde{x}^d \} \\
S_2 &:= \{ e : e_4 \ne 0 \text{ and } \tilde{e}_1 = x^* - \tilde{x}^d \} \\
S_3 &:= \{ e : e_4 = 0 \text{ and } \tilde{e}_1 = x^* - \tilde{x}^d \}
\end{aligned} \tag{22}$$

Since the mathematical form of the discontinuity surfaces is similar to those shown in Section 4.1, the procedure for proving the unique Filippov's solution on the intersecting discontinuity surface is also the same as the one shown in Section 4.1. For the sake of brevity, only the results are presented here. The results from our program are summarized in Table 2, which show that there are 243 cases for determining the intersections. Among them, 96 cases are removed due to the constraints imposed on the variables. Among the 147 possible cases, 81 cases contain one and only one non-empty set. Thus, if the hydraulic actuator systems operate in the regions described by the above 81 cases, the systems have unique Filippov's solution. However, for the remaining 66 cases, more than one set is non-empty. Thus our method is not conclusive on the uniqueness of Filippov's solution for these 66 cases.

Although the uniqueness of Filippov's solutions can not be concluded for the 66 cases, our method identified all the non-empty sets for each case. Two approaches are recommended. One is that the 66 cases should be investigated using Filippov's original theorem. The second approach is to design the controller, which avoids the occurrence of the cases where the uniqueness of solution is questionable [11]. Regardless of the approach, our method provides not only crucial information about the uniqueness of Filippov's solution, but also important guidelines for control design to guarantee a unique Filippov's solution.

Table 2 Summary of the results for Case Study 2.

Information	Number of cases
Total cases	243
Impossible cases	96
One and only one non-empty set	81
More than one non-empty set	66

5. CONCLUSIONS

Many physical systems, such as robotic control systems, have been modeled as non-smooth systems. Non-smooth systems often have more than one solution. Thus, it is important to ensure and to prove the non-smooth mathematical model of the physical systems to have only one solution. Otherwise, under what conditions and which solution will show up become crucial issues [9]. However, such a proof has been extremely challenging especially when the discontinuity surface is the intersection of multiple discontinuity surfaces. This is due to the requirement of determining the intersections of the sets from Filippov's inclusions and their associated sets containing vectors tangent to the discontinuity surfaces [1,2]. Therefore, although many physical systems have been modeled as non-smooth systems, proper solution analysis has been extremely limited.

This paper presented a method to facilitate the analysis of the uniqueness of Filippov's solutions to non-smooth systems when the discontinuity surface is the intersection of multiple discontinuity surfaces. Two examples of practical non-smooth control systems were presented to demonstrate the efficacy of the method. The proposed method can not only prove the unique Filippov's solution for a class of non-smooth systems, but also output explicitly the emptiness/non-emptiness of each set in each region. Such information is crucial in numerical simulation since it indicates the surface that the solution trajectory moves to. Thus the work contributes to the analysis of non-smooth systems where the proof of the uniqueness of Filippov's solution is crucial to keep the mathematical model relevant to physical systems [12] and to ensure the numerical solutions can be sought [3].

The main limitation of the method, developed here, is that, due to the expansion of sets, the method is more restrictive as compared to Filippov's original theorem. Although for some cases, our method is not conclusive, our method identifies all the non-empty sets. Thus, two approaches are recommended. One is to investigate the unique Filippov's solution for the non-conclusive cases by Filippov's original theorem. Another one is to design controllers to prevent the solution trajectories from entering the regions where the uniqueness of Filippov's solutions is questionable [11]. Future work includes different expansions of the sets to make the method less restrictive. Although the work is preliminary, it is an important stepping stone to fill the gap of analysis of unique Filippov's solution of many practical non-smooth robotic control systems regarding the application of Filippov's solution theory.

6. ACKNOWLEDGEMENT

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