A REVIEW OF FORMULAE FOR AVERAGING PHYSICAL QUANTITIES (APPLICATION TO CALCULATION OF THE AVERAGE RADIUS OF TUBES)

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Received June 2006, Accepted January 2007
No. 06-CSME-30, E.I.C. Accession 2949

ABSTRACT

A new method for averaging physical quantities is discovered. It is shown that the traditional method of finding the average value of a physical quantity gives the wrong results when calculating the average radius of a tapering tube, the average flow velocity in the tube and the volume of liquid flow through the tapering tube. The new method of averaging gives the correct results. The new formula is applicable to many other processes, for example, for calculating the flow through tubes of arbitrary form or with time-dependent radius. At present, a neutral radius is used which leads to big discrepancies.

Keywords: Liquid flow in tubes; Elastic tubes; Averaging physical quantities; Pulsatile flow; Blood flow
PACS: 47.27.nd; 47.27.nf; 47.60.+i; 87.19.Uv; 47.55.dr; 83.50.Ha

REVUE DE FORMULES POUR MOYENNER DES QUANTITÉS PHYSIQUES (APPLICATION AU CALCUL DU RAYON MOYEN DE TUBES)

RESUME

Une nouvelle méthode pour moyenner des quantités physiques a été découverte. Il a été montré que la méthode traditionnelle de détermination de valeur moyenne de quantité physique, donne de mauvais résultats pour le calcul de rayon de tube conique, la vélocité moyenne du flux dans le tube ainsi que le volume de flux liquide dans le tube conique. La nouvelle méthode de moyennage donne des résultats correctes. la nouvelle formule est applicable a beaucoup d'autres processus comme par exemple, le calcul de flux dans des tubes de formes arbitraires, ou avec des rayons variant avec le temps. En ce moment, nous utilisons un rayon neutre qui mène à d'importantes contradictions.

INTRODUCTION

In some problems of engineering and biology, flow in tapering tubes is considered, for example the flow of blood in vessels. Sometimes it is useful to find the average radius of a tapering tube or, in general, that of a tube with variable radius. Consider the flow of liquid in a tube which has the form of a truncated cone. Its left broad side has radius R_1 and its right narrow side has radius R_2 and its length is L. It is necessary to find the average radius of the tube. The traditional equation of finding the average value reads as

$$\overline{f(x)} = 1/(x_2 - x_1) \int_{x_1}^{x_2} f(x)dx . \tag{1}$$

It is the only method of averaging functions (Korn and Korn, 1968). If in this equation the variable x is the time then it provides temporal averaging. If the variable x is a spatial coordinate at an instant in time then it provides spatial averaging. By Eq. 1, the average radius $\bar{R}$ is equal to $(R_1 + R_2)/2$. If $R_2 = 0$, then the flow through the tube is impossible, but Eq. (1) gives $\bar{R} = R_1/2$. The flow through the tube is proportional to the square of the average radius, therefore, according to Eq. (1), flow in the tube must exist. It is a contradiction, and one sees that Eq. (1) is not applicable in this case. In this paper another equation of averaging is proposed which gives the true results.

THEORY

In the physics of friction there is the following problem. It is necessary to find the average width of a clearance with roughness when lubrication flows through this clearance. The size of roughness is comparable to the width of the clearance.

Let us consider a two-dimensional clearance. Roughness is present on its lower and upper sides. The width of the clearance is denoted by h(x). The x axis is directed along the clearance. One can write

$$h(x) = h_0(x) + \varepsilon_1(x) + \varepsilon_2(x) = h_0(x) + \varepsilon(x) \tag{2}$$

where $h_0(x)$ is the width of the clearance without roughness, $\varepsilon_1(x)$ and $\varepsilon_2(x)$ are random functions which describe the roughness on the lower and upper sides, respectively, and $\varepsilon(x) = \varepsilon_1(x) + \varepsilon_2(x)$. It was assumed that $\langle\varepsilon(x)\rangle = 0$ and $\langle h(x)\rangle = \langle h_0(x)\rangle$ (Galahov and Usov 1990; Usov 1983, 1984, 1986). Averaging in (Galahov and Usov 1990; Usov 1983, 1984, 1986) was done by Eq. (1). According to Eq. (1), $\langle h(x)\rangle$ always equals $\langle h_0(x)\rangle$ and is independent of the size of roughness. However, if $\varepsilon(x)$ equals $h_0(x)$ then liquid can not flow through the clearance. It is necessary to find a new method of averaging.

The author solved this problem as follows. Consider the following model problem. A ship goes along a river from the point A to the point B and back. Its speed in still water is v_0 and the speed of the flow is a. The distance from A to B is L. It is necessary to find the average speed of the ship during the travel from A to B and back. According to definition, the mean value of the function f(x) at the interval $[x_1, x_2]$ is given by Eq. (1) (Korn and Korn, 1968). Therefore, one can strongly apply Eq. (1) for finding the average speed. This equation gives the average speed equal to v_0, but this solution is wrong:

$$\langle v\rangle = 1 / (2L) \int_0^{2L} v(x)dx = 1 / (2L) \left(\int_0^{L} (v_0 + a)dx + \int_L^{2L} (v_0 - a)dx \right) = v_0. \tag{3}$$

One can make a conclusion: the traditional method of averaging physical quantities is wrong for some processes.

The correct solution is given by the equation

$$\langle v\rangle = 2L / (L / (v_0 + a) + L / (v_0 - a)) = v_0 - a^2 / v_0. \tag{4}$$

One can find the width of the clearance with roughness analogously. Let us suppose that the average roughness height is a / 2. Then the sum of the average roughness heights of the lower and upper sides is a, and

$$\langle h(x)\rangle = \langle h_0(x) + \varepsilon(x)\rangle = L / (L / (2(h_0(x) - a)) + L / (2(h_0(x) + a))) = h_0(x) - a^2 / h_0(x). \tag{5}$$

Here it is taken into account that half of the clearance length is occupied by the peaks, and the other half is occupied by the cavities. One sees that if a changes from $0.1h_0(x)$ to $h_0(x)$, then $\langle h(x)\rangle$ changes from $0.99h_0(x)$ to 0.

The general equation of finding the average speed of the ship will be

$$\langle v\rangle = 2L / \int_0^{2L} dx / v(x), \tag{6}$$

and the general equation of the new type of averaging will be

$$\langle f(x)\rangle = (x_2 - x_1) / \int_{x_1}^{x_2} dx / f(x). \tag{7}$$

Eq. (6) provides a sort of inverse spatial average of the function 1 / v(x) (time is not constant).

Although dx / v(x) = dt and Eq. (6) can be formally transformed to a temporal average $2L / \int_{t_1}^{t_2} dt$

where $t_1 = 0$ and $t_2 = \int_0^{2L} dx / v(x)$, the new equation consists of two integrals in the denominator and primarily Eq. (6) is an inverse spatial average.

If in Eq. (7) the variable x does not depend on the time then it is the inverse spatial averaging of the function 1 / f(x). If the variable x does depend on the time then the interpretation of this equation is not so straightforward. If the variable x is the time then, as it will be shown later, this equation has no apparent physical sense.

Now let us use Eq. (7) for calculation of the average radius of a tapering tube. The dependence of the radius on the distance along the axis of the tube is $R(x) = (R_2 - R_1) / L \cdot x + R_1$, $x_2 - x_1 = L$ hence

$$\langle R \rangle = (R_2 - R_1) / (\ln R_2 / R_1). \tag{8}$$

Eq. (7) can be used in engineering for calculation of mean radius of a tube with $R_1 \gg R_2$ or in biology for calculation of the mean radius of a blood vessel whose end is almost occluded.

It is necessary to check the validity of Eq. (7). One can calculate the average velocity of liquid in the tube by Eqs. (1) and (7). According to the definition, the average velocity is the velocity whose displacement for the time t is equal to the displacement for the variable velocity for the same time:

$$\langle v \rangle t = L. \tag{9}$$

Using the continuity equation

$$R_1^2 v_1 = R_2^2 v_2 \tag{10}$$

(v_1 and v_2 are the fluid velocities at the inlet and at the outlet of the tube, respectively) one can calculate $\bar{v}$ by Eq. (1) and $\langle v \rangle$ by Eq. (7):

$$\bar{v} = R_1 / R_2 v_1 \tag{11}$$

and

$$\langle v \rangle = 3 R_1^2 v_1 (R_2 - R_1) / (R_2^3 - R_1^3). \tag{12}$$

The time t can be found from the equation

$$t = \int_0^L dx / v(x), \tag{13}$$

namely

$$t = L(R_2^3 - R_1^3) / 3R_1^2 v_1(R_2 - R_1). \tag{14}$$

It is clear that $\bar{v}t \neq L$ but $\langle v \rangle t \equiv L$.

Another simple example from inviscid fluids: Bernoulli's equation reads

$$P_1 + 0.5\rho v_1^2 = P_2 + 0.5\rho v_2^2 \tag{15}$$

where ρ is the density of the liquid. Using it one can calculate $\bar{v}$ from Eq. (1) and $\langle v \rangle$ from Eq. (7). One can easily show that $\bar{v}t \neq L$. However,

$$\langle v \rangle = L / \int_0^L dx / v(x) \tag{16}$$

and using Eq. (13) it is clear that $\langle v \rangle t \equiv L$.

Let us solve the following problem. It is necessary to find the volume of liquid flow through a tapering tube. For a tube with constant radius, the Poiseuille's equation gives

$$Q = \Pi R^4(P_1 - P_2) / (8\mu L), \tag{17}$$

where μ is the dynamic viscosity of the liquid. One can try to calculate the volume flow through the tapering tube by the following formula:

$$Q = \Pi \langle R^4 \rangle (P_1 - P_2) / (8\mu L) \tag{18}$$

where $\langle R^4 \rangle$ is the average value of R^4. One can show that averaging using of Eq. (1) leads to a wrong result. Averaging using Eq. (7) gives

$$\langle R^4 \rangle = L / \int_0^L dx / R^4 = 3(R_1 - R_2) / (1/R_2^3 - 1/R_1^3). \tag{19}$$

Introducing it into Eq. (18), one gets

$$Q = 3\Pi(R_1 - R_2)(P_1 - P_2) / (8\mu L(1/R_2^3 - 1/R_1^3)). \tag{20}$$

One can find a few exact solutions to this problem using differential calculus. One of them is to treat a thin tapering liquid shell of thickness dr, inner radii r_1 at the left edge and r_2 at the right edge, and length L; its axis is coincident with the axis of the tube, then to consider pressure and viscous forces acting on it. The simplest method is: for infinitely small element of tube with length dx, Eq. (17) can be used if $(P_1 - P_2) / L$ is replaced by $-dP / dx$:

$$Q = - \Pi R^4 dP / (8\mu dx). \tag{21}$$

Then

$$- dP = 8\mu Q / (\Pi R^4)dx \tag{22}$$

and

$$P_1 - P_2 = 8\mu Q / \Pi \int_0^L dx / R^4 = 8\mu Q / (3\Pi)L / (R_1 - R_2)(1 / R_2^3 - 1 / R_1^3). \tag{23}$$

From Eq. (23), Eq. (20) follows. One sees that the exact method is more protracted.

One can try to use this method for calculation of the average radius of elastic tubes with diameter varying with time (for example, pulsatile flow in blood vessels) and for simple calculation of the volume of liquid flow through such tubes. In previous calculations a neutral radius was used. It leads to big discrepancies for large radial movements (Zamir 2000).

When the tube is nonrigid, the fluid velocity depends on x and t: v(x, t) and the flow rate is

$$Q(x, t) = \int_0^{R_0} 2\Pi R v(x, t)dR \tag{24}$$

where R_0 is a neutral radius of the tube (Zamir 2000). It is necessary to replace R_0 by $\langle R \rangle$ found by Eq. (7). If $v(x, t) = 1$ then $Q(x, t) = \Pi \langle R \rangle^2 = S$. Therefore,

$$\langle R \rangle = \langle S / \Pi. \rangle^{0.5} = [(x_2 - x_1) / \int_{x_1}^{x_2} dx / R(x,t)^2]^{0.5} =$$

$$[L / \int_0^L dx / R(x,t)^2]^{0.5} \tag{25}$$

where L is the length of the tube. In the derivation of v(x, t) differential equations are solved with the boundary condition R_0 (Zamir 2000). It is necessary to change this boundary condition to $\langle R \rangle$.

For a nonrigid tube with $P_1 = P_0 + P' \cos(\omega t)$ and constant P_0, P' and P_2, Eq. (18) will turn to

$$Q(t) = \Pi \langle R(x, t)^4 \rangle (P_0 + P' \cos(\omega t) - P_2) / (8\mu L). \tag{26}$$

CONCLUSIONS

One can make the following conclusions. A new formula for averaging physical quantities is discovered: Eq. (7). The traditional equation of averaging (1) gives the wrong results for the calculation of the average values of physical quantities in tubes with a flow of liquid. The proposed equation (7) gives the correct results. The probable reason is that Eq. (1) does not take into account interaction of liquid with the walls of the tube. It is shown that Eq. (1) is not the universal one as it was supposed earlier.

The new method of averaging has limitations: it seems, it is not valid for time averaging ($\overline{f(t)}(t_2 - t_1) = \int_{t_1}^{t_2} f(t)dt$ has physical sense but $<f(t)>(t_2 - t_1) = (t_2 - t_1)^2 / \int_{t_1}^{t_2} dt / f(t)$ has no apparent physical sense), and for cases when $\int_{x_1}^{x_2} dx / f(x) = 0$.

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NOMENCLATURE

The following symbols are used in this Technical Note:

a = 1. speed of the flow of river 2. twice the the average roughness height;
f(x) = function of the argument x;
h(x) = the width of the clearance;
$h_0(x)$ = the width of the clearance without roughness;
L = length;
r_1 = inner radius of thin tapering liquid shell at the left edge;
r_2 = inner radius of thin tapering liquid shell at the right edge;
dr = thickness of thin tapering liquid shell at the right edge;
R_1 = radius of truncated cone at the left side;
R_2 = radius of truncated cone at the right side;

$R(x, t)$ = radius as a function of coordinate and time;

$\bar{R}$ = average radius averaged by equation (1);
<R>= average radius averaged by equation (7);
v = velocity;
v_0 = the speed of ship in still water;
v_1 = fluid velocity at the inlet a of the tube;
v_2 = fluid velocity at the outlet of the tube;
$v(x)$ = velocity of liquid as a function of coordinate;

$\bar{v}$ = average velocity averaged by equation (1);
<v> = average velocity averaged by equation (7);
t = time
P_1 = pressure at the left side of truncated cone;
P_2 = pressure at the right side of truncated cone;
Q = volume of liquid flow through a tube;
x = running coordinate;

$\varepsilon_1(x)$ = random function which describes the roughness on the lower;
$\varepsilon_2(x)$ = random function which describes the roughness on the upper side;
$\varepsilon(x) = \varepsilon_1(x) + \varepsilon_2(x)$;
μ = the dynamic viscosity of liquid;
$\Pi = 3.14159$;
ρ = the density of liquid.
ω = angular frequency