SINGULARITIES OF REDUNDANT 4R POSITIONING MANIPULATORS

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Received June 2007, Accepted November 2007
No. 07-CSME-50, E.I.C. Accession 3019

ABSTRACT

Singularities can be elusive but geometric considerations can reveal the singularities of a redundant 4R manipulator for positioning tasks. Points on a line, called *transversal*, that intersects all R-joint axes, cannot move in the direction of this line. Conditions governing the existence of transversals to four lines will be discussed. A way to find transversals is developed and tested with a numerical example. A possible metric, or singularity proximity measure for this type of singularity is investigated. This metric is based on the shortest distance between a line and a quadric and various methods will be proposed to solve this geometric problem.

Keywords: singularity, 4-R manipulator, kinematic geometry

SINGULARITÉS D'UN MANIPULATEUR REDONDANT DE POSITIONNEMENT À 4R

RÉSUMÉ

Les singularités sont parfois insaisissables, mais une approche géométrique permet de découvrir certaines singularités d'un manipulateur redondant de positionnement à 4R. Tous les points appartenant à une ligne, nommée transversal, qui intercepte tous les axes de rotation du manipulateur, ne peuvent se déplacer dans la direction prescrite par cette ligne. Les conditions qui régissent l'existence des *transversals* à quatre lignes seront discutées. De plus, une méthode pour les calculer sera proposée et vérifiée avec un exemple numérique. Finalement, un métrique, ou une mesure de proximité, associé à cette singularité sera étudié. Ce métrique est basé sur la plus courte distance entre une ligne et une surface quadratique et plusieurs méthodes seront proposées pour résoudre ce problème géométrique.

Mots clés: singularité, manipulateur 4-R, géométrie cinématique

1 INTRODUCTION

Singularity in a serial manipulator represents a loss of mobility. Therefore singularity detection and mapping is important in planning trajectories. This paper focuses on a geometric methodology, that clearly illustrates positioning singularities in a four revolute (4R) serial manipulator. First, the geometric nature of the singularities of interest will be studied. This will reveal how it applies to 4R manipulators and will lead to the formulation of the problem of finding transversals to four lines. After introducing some peculiar but useful mathematical notation, a problem solution will be outlined and various possible transversal configurations will be studied in terms of their effect on restricting the positioning workspace.

2 QUALITATIVE DESCRIPTION OF THE SINGULARITY

2.1 Three Degrees-of-Freedom Serial Positioners

Consider a serial chain with three joints.

- Given three prismatic or P-joints, there are no singularities, save those imposed by joint limit of travel, if the joint axes, lines on the absolute plane at infinity $\omega\{1:0:0:0\}$, do not intersect on a common point.
- Given three revolute or R-joints with axes, lines in Euclidean space, having no points of intersection, there will be a two-parameter set of points on the end effector (EE) whose mobility will be restricted. This set is described by the quadric surface, usually a hyperboloid of one sheet or a hyperbolic paraboloid, produced by the one-parameter set of lines ruling the three given ones. No point on the surface may move along the ruling line that contains it.

2.2 Positioning and the "Operating Point"

It is conventional, when describing robotic manipulation, to define and speak of a particular point, usually the origin of a Cartesian frame attached to EE, as the *operating point* (OP). In a positioning manipulator the OP is generally destined to be placed at desired locations described in the base or fixed frame (FF). In all that follows, singularity-induced loss of mobility will be described in terms of forbidden directions of motion in some subset of the two-parameter line bundle, a special linear congruence, on OP. *I.e.*, the EE is, from a geometric point of view, an infinite rigid body that occupies all of space and "moves". Some lines in this space contain points that cannot be moved along them. Singularity, with respect to OP, occurs if that congruence contains any of these lines.

2.3 Singularity of 4R Manipulators

The positioning singularities of a 4R manipulator can be best understood by considering the single degree-of-freedom (1 dof) system of a single revolute joint (R-joint). Any point on any line that intersects an R-joint axis will move on a circular trajectory and its velocity vector will always be perpendicular to the line. Therefore, it is obvious that these points cannot move in the direction of the line.

Consider now an open or serial kinematic chain of R-joints. The singularity described above appears if there exist lines that intersect every R-joint axis. All the lines in space constitute a four-parameter set (Hunt [1]). Therefore, the system of lines that is constrained to intersect any other given one represents a set of ∞^3 lines; a 3 dof set called a *special linear line complex.* Imposing two or three given lines to intersect the set, the system is reduced to ∞^2 or ∞^1 lines, respectively. These configurations are respectively called *congruence* and *line series*. The 1 dof set is commonly known as a ruled surface. Finally, the system of lines that intersects four given ones represents ∞^0 lines, which is an integer. This number will be investigated from a geometrical point of view in Section 3.1, but the existence of a finite number of solutions explains why a 4R manipulator cannot avoid possible singular configurations and therefore warrants investigation.

3 TRANSVERSALS TO FOUR LINES

Transversals to four lines defined by the 4R-joint axes identify directions of forbidden translation of certain points on the distal link or end effector (EE). The problem of finding those transversals will be formulated algebraically, but it also has a very interesting geometric significance that will be explained now.

3.1 Qualitative Approach to the Transversal Problem

In general, a line with 1 dof generates a ruled surface. For example, a line ruling three fixed lines generates a quadric surface, *i.e.*, a hyperboloid of one sheet, a hyperbolic paraboloid or, in particular degenerate cases, a pair of planes. Such quadrics are doubly ruled, *i.e.*, they have two sets of generators, one called *regulus* and the other one, *complementary regulus.* A generator in one regulus intersects all generators of the complementary regulus. Three fixed, generally skew, lines defining the quadric belong to the same regulus. The moving or ruling line belongs to the complementary one. Ruled surfaces in kinematics are treated in detail by Hunt [1].

The 4R manipulator joint axes define four lines of which any set of three can be selected to generate a quadric surface. Then the remaining line can either pierce the quadric surface in two different points, be tangent to it or simply have no real intersection. If the line pierces the quadric, at each of the intersection points, two generators intersect it. One generator belongs to the regulus of the three given lines but the other one belongs to the complementary regulus. The latter therefore intersects the four lines and represents a transversal. Hence, there are in general two transversals to four lines. If the fourth line is tangent to the quadric, a double line is obtained. However, if the fourth line does not intersect the quadric, the transversals are complex and there is no translational singularity for this pose. Using the distance between the fourth line and the quadric as a metric to describe "closeness" to singularity will be examined in Section 4.2.

3.2 Mathematical Notation

The mathematical notation will be introduced now. A point X will be designated with four homogeneous coordinates, *i.e.*, $X\{x_0 : x_1 : x_2 : x_3\}$. When $x_0 = 1$, the point is in Euclidean space, while $x_0 = 0$ represents a point on the absolute plane at infinity. A point X on plane a must satisfy the plane equation $A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3 = 0$. Plane coefficients are called homogeneous plane coordinates, $a\{A_0 : A_1 : A_2 : A_3\}$.

Lines are defined by their six homogeneous Plücker coordinates. A line in space can be defined with two points or two planes. Depending on which are used, the coordinates are called radial (for points) or axial (for planes). A line C, defined by points X and Y, has coordinates $C_r\{c_{01} : c_{02} :$

$c_{03} : c_{23} : c_{31} : c_{12}\}$ where the six possible 2×2 determinants of a 2×4 matrix of homogeneous point coordinates produce the line coordinates thus:

$$c_{ij} = \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}, \quad i, j = 0, 1, 2, 3 \tag{1}$$

An axial line is defined by planes a and b. These line coordinates are denoted in upper case, $\mathcal{C}_a\{C_{01} : C_{02} : C_{03} : C_{23} : C_{31} : C_{12}\}$. In Eq. (1) the point coordinates are replaced by respective plane coordinates. Both formulations may represent the same line. In such cases $[c_{01} : c_{02} : c_{03}] = \lambda[C_{23} : C_{31} : C_{12}]$ represents the direction of the line and $[c_{23} : c_{31} : c_{12}] = \lambda[C_{01} : C_{02} : C_{03}]$ represents its moment about the origin. Note that (:) indicates proportionality of homogeneous coordinates hence the inclusion of the arbitrary non-zero multiplier λ. Six numbers represent a line only if the direction vector is normal to the moment vector. This is called the Plücker condition and it is expressed by Eq. (2). Both Hunt [1] and Klein [2] deal with line geometry.

$$c_{01}c_{23} + c_{02}c_{31} + c_{03}c_{12} = C_{01}C_{23} + C_{02}C_{31} + C_{03}C_{12} = 0 \tag{2}$$

3.3 The Congruence of Lines on a Point

To illustrate a simple exercise in line geometry before introducing the somewhat more involved notion of the hyperboloid on three lines in Section 3.4, below, consider the congruence referred to above in Sections 2.2 and 2.3. A line $\mathcal{G}_a$ and a point P not on it define a plane p. The coefficients or plane coordinates are P_i as given below.

$$P_i = \sum_{j=0}^{3} G_{ij}p_j, \quad G_{ii} = 0, \quad G_{ji} = -G_{ij}, \quad i = 0, 1, 2, 3 \tag{3}$$

If $P \in \mathcal{G}$ then $P_i = 0, \ \forall \ i$. Expanding the expression above, any two of the four linearly dependent equations below may be used as constraints to define the congruence of a line bundle $\mathcal{G}$ on P.

$$\begin{bmatrix} 0 & G_{01} & G_{02} & G_{03} \\ -G_{01} & 0 & G_{12} & -G_{31} \\ -G_{02} & -G_{12} & 0 & G_{23} \\ -G_{03} & G_{31} & -G_{23} & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

3.4 Quadric Surface Defined by Three Lines

As explained in Section 3.1, the existence of real transversals to four given lines depends on the type of intersection that one line has with the quadric defined by the three others. In this section, the point equation of the quadric described by three lines will be obtained in terms of its line coordinates. The desired surface is obtained by the motion of line $\mathcal{M}$ constrained to intersect the three given lines $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$. Based on Pottmann et al. [3], these intersection conditions are readily written as:

$$P_{01}m_{01} + P_{02}m_{02} + P_{03}m_{03} + P_{23}m_{23} + P_{31}m_{31} + P_{12}m_{12} = 0 \tag{5}$$

$$Q_{01}m_{01} + Q_{02}m_{02} + Q_{03}m_{03} + Q_{23}m_{23} + Q_{31}m_{31} + Q_{12}m_{12} = 0 \tag{6}$$

$$R_{01}m_{01} + R_{02}m_{02} + R_{03}m_{03} + R_{23}m_{23} + R_{31}m_{31} + R_{12}m_{12} = 0 \tag{7}$$

The quadric surface can be expressed in point coordinates, rather than in the coordinates of line $\mathcal{M}$, using the condition stating that a point X is on line $\mathcal{M}$, namely, (see Hodge and Pedoe [4] and Sommerville [5])

$$m_{23}x_1 + m_{31}x_2 + m_{12}x_3 = 0 \quad (8)$$

$$-m_{23}x_0 + m_{03}x_2 - m_{02}x_3 = 0 \quad (9)$$

$$-m_{31}x_0 - m_{03}x_1 + m_{01}x_3 = 0 \quad (10)$$

$$-m_{12}x_0 + m_{02}x_1 - m_{01}x_2 = 0 \quad (11)$$

From this set of four equations, only two are independent. With Eqs. (5-7, 9,10), one can form the following system of linear equations.

$$\begin{bmatrix} P_{01} & P_{02} & P_{03} & P_{23} & P_{31} & P_{12} \\ Q_{01} & Q_{02} & Q_{03} & Q_{23} & Q_{31} & Q_{12} \\ R_{01} & R_{02} & R_{03} & R_{23} & R_{31} & R_{12} \\ 0 & -x_3 & x_2 & -x_0 & 0 & 0 \\ x_3 & 0 & -x_1 & 0 & -x_0 & 0 \end{bmatrix} \begin{bmatrix} m_{01} \\ m_{02} \\ m_{03} \\ m_{23} \\ m_{31} \\ m_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (12)$$

Because the line coefficients m_{ij} are homogeneous, the system of Eq. (12) can be solved by setting any of them to unity. Choosing $m_{12} = 1$, Eq. (12) becomes:

$$\begin{bmatrix} P_{01} & P_{02} & P_{03} & P_{23} & P_{31} & P_{12} \\ Q_{01} & Q_{02} & Q_{03} & Q_{23} & Q_{31} & Q_{12} \\ R_{01} & R_{02} & R_{03} & R_{23} & R_{31} & R_{12} \\ 0 & -x_3 & x_2 & -x_0 & 0 & 0 \\ x_3 & 0 & -x_1 & 0 & -x_0 & 0 \end{bmatrix} \begin{bmatrix} m_{01} \\ m_{02} \\ m_{03} \\ m_{23} \\ m_{31} \end{bmatrix} = \begin{bmatrix} -P_{12} \\ -Q_{12} \\ -R_{12} \\ 0 \\ 0 \end{bmatrix} \quad (13)$$

Expressions for m_{ij} are lengthy sums of products that are not expanded here. To obtain the equation of the quadric surface, these expressions for line $\mathcal{M}$ are substituted into Eq. (2) as

$$m_{01}m_{23} + m_{02}m_{31} + m_{03}m_{12} = 0 \quad (14)$$

Eq. (14) factors. One factor is a conic containing only x_0 and x_3. The other is the expected quadric surface. Its 10 coefficients a_{ij} appear in Eq. (15) below:

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad (15)$$

3.5 Finding the Transversals

Here, a method to find the transversals $\mathcal{T}$ to four given lines ($\mathcal{P}$,$\mathcal{Q}$,$\mathcal{R}$ and $\mathcal{S}$) will be described. The four intersection conditions and the Plücker condition for line $\mathcal{T}$ gives the following five equations.

$$P_{01}t_{01} + P_{02}t_{02} + P_{03}t_{03} + P_{23}t_{23} + P_{31}t_{31} + P_{12}t_{12} = 0 \quad (16)$$
$$Q_{01}t_{01} + Q_{02}t_{02} + Q_{03}t_{03} + Q_{23}t_{23} + Q_{31}t_{31} + Q_{12}t_{12} = 0 \quad (17)$$
$$R_{01}t_{01} + R_{02}t_{02} + R_{03}t_{03} + R_{23}t_{23} + R_{31}t_{31} + R_{12}t_{12} = 0 \quad (18)$$
$$S_{01}t_{01} + S_{02}t_{02} + S_{03}t_{03} + S_{23}t_{23} + S_{31}t_{31} + S_{12}t_{12} = 0 \quad (19)$$
$$t_{01}t_{23} + t_{02}t_{31} + t_{03}t_{12} = 0 \quad (20)$$

As an aside, it is interesting to note that Eqs. (16-18) are similar to Eqs. (5-7) of the previous section. The above system of equations clearly demonstrates what was stated in Section 3.1, *i.e.*, real transversals occur only if the fourth line intersects the quadric defined by the three other lines. It can also be seen here that the choice of the three lines, among the four available, used to generate the hyperboloid is arbitrary and has no influence on the resulting transversals. To continue, an expression of t_{01}, t_{02}, t_{03} and t_{23} in terms of t_{31} and t_{12} can be obtained by solving the following system of equations.

$$\begin{bmatrix} P_{01} & P_{02} & P_{03} & P_{23} \\ Q_{01} & Q_{02} & Q_{03} & Q_{23} \\ R_{01} & R_{02} & R_{03} & R_{23} \\ S_{01} & S_{02} & S_{03} & S_{23} \end{bmatrix} \begin{bmatrix} t_{01} \\ t_{02} \\ t_{03} \\ t_{23} \end{bmatrix} = \begin{bmatrix} -P_{31}t_{31} - P_{12}t_{12} \\ -Q_{31}t_{31} - Q_{12}t_{12} \\ -R_{31}t_{31} - R_{12}t_{12} \\ -S_{31}t_{31} - S_{12}t_{12} \end{bmatrix} \quad (21)$$

Expressions for t_{01}, t_{02}, t_{03} and t_{23}, like those for m_{ij}, are too long to write out. Substituting those results in the Plücker condition, Eq. (19), gives a second order bivariate of the form

$$At_{31}^2 + Bt_{31}t_{12} + Ct_{12}^2 = 0 \quad (22)$$

where A, B and C are also too long to write here. Because the line coefficients t_{ij} are homogeneous, Eq. (22) can be reduced to an univariate by setting t_{31} or t_{12} to unity. A similar expression was obtained by Teller and Hohmeyer [6]. This may have two real solutions, a double one or two complex roots. Then, the other line coordinates (t_{01},t_{02},t_{03} and t_{23}) are readily calculated by back-substitution using Eq. (21).

3.6 Numerical Example

In this section, the formulæ already derived will be used in a numerical example. Consider the 4R manipulator illustrated in Figure 1. The line coordinates of the four axes of rotation and the position of the joints are given in Table 1. The radial coordinates of the two transversals, as outlined in Section 3.5, also appear there.

With the four lines, four different quadrics can be generated. They are illustrated in Figure 2. One can see that the fourth line, which appears as a solid line, always pierces the quadric even if the latter depends on the set of lines selected to generate the surface. The two other lines illustrated are the two transversals, $\mathcal{T}_1$ and $\mathcal{T}_2$, plotted as dashed and dashed-dotted lines respectively.

Table 1: Line and joint coordinates for the manipulator of Figure 1.

Line	Joint position
$\mathcal{P}_a\{0:-2:0:0:0:1\}$	$P\{1:2:0:0\}$
$\mathcal{Q}_a\{-3.7648:1:2.2409:-0.6322:0.1115:-1.1119\}$	$Q\{1:1.1206:1.3160:1\}$
$\mathcal{R}_a\{-0.0616:1:-0.2226:0.1124:-0.0389:-0.2060\}$	$R\{1:3.7638:0.6771:2\}$
$\mathcal{S}_a\{-1.9337:1:1.1599:-0.1627:0.1938:-0.4383\}$	$S\{1:3.3949:3.0851:3\}$
$\mathcal{T}_1\{-0.0001:0.2888:-0.5003:-2.1556:1:0.5776\}$	
$\mathcal{T}_2\{-0.2634:-0.1975:-0.5704:0.1056:1:-0.3951\}$	

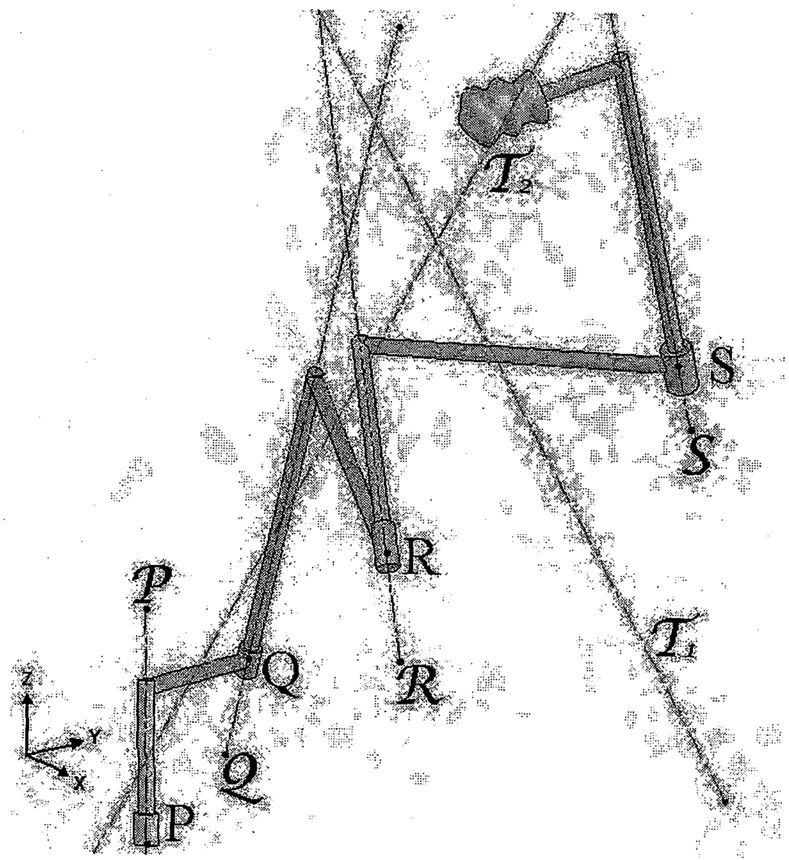
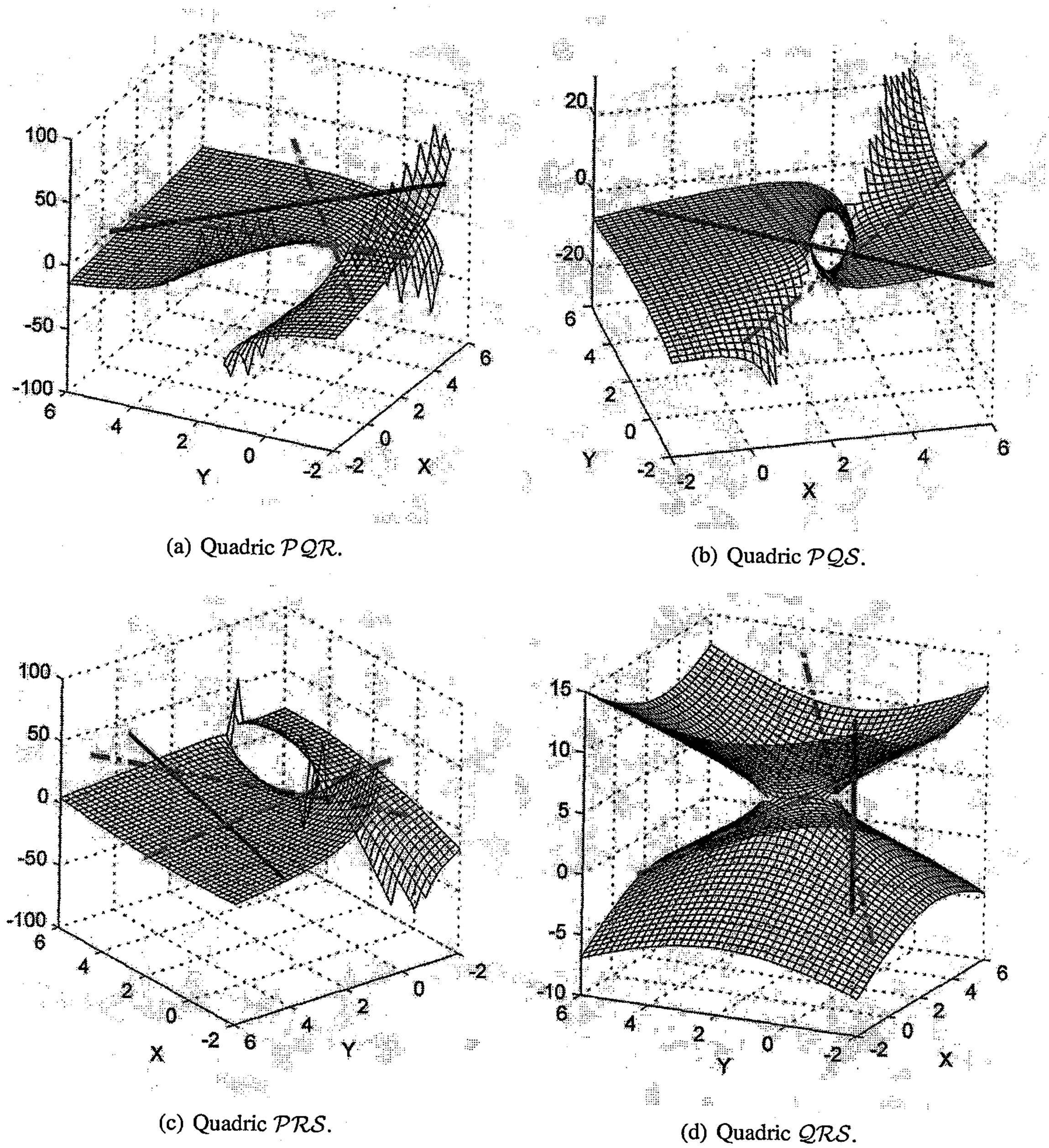


Figure 1: Example of a 4R manipulator.

The transversals also appear in Figure 1. One can see that $\mathcal{T}_2$ intersects the end-effector. Therefore, all points on the EE that are on $\mathcal{T}_2$ will not be able to move in the direction of $\mathcal{T}_2$. On the other hand, $\mathcal{T}_1$ does not intersect the end effector; therefore no singularity is associated with this line.



(a) Quadric $\mathcal{PQR}$.

(b) Quadric $\mathcal{PQS}$.

(c) Quadric $\mathcal{PRS}$.

(d) Quadric $\mathcal{QRS}$.

Figure 2: Four quadric possibilities; their remaining line and two transversals.

4 Possible Use of a Metric

As explained in Section 2, real transversals exist only if one of the four given lines intersects the quadric surface ruled by the three others. Therefore, if that line does not intersect the quadric, the shortest distance between those two elements might be used as a metric. To devise a figure of merit in kinematics is often a quixotic enterprize because solution space, even if it is a metric one, bears little or no significant relation to the space of Euclidean motion. Here the situation is satisfyingly better. All four lines are Euclidean as is the quadric so measured distance is real. For this reason it was believed that distance between line and quadric might represent a useful, possibly good, way to assess proximity to singularity. A procedure to find this distance will be described and a numerical example will be used to study this metric and to reveal a serious flaw.

4.1 Shortest Distance Between a Quadric and a Line

Consider a remaining line $\mathcal{S}$ that has no real points on the quadric given by Eq. (15). The point X on the quadric and closest to line $\mathcal{S}$ has to satisfy the following conditions:

1. The normal to the quadric surface on point X is perpendicular to line $\mathcal{S}$.
2. The line normal to the quadric on point X has to intersect line $\mathcal{S}$.

These conditions are easy to formulate mathematically but result in very lengthy expressions. The coordinates of plane n tangent to the quadric surface on point $X\{1 : x_1 : x_2 : x_3\}$ can be obtained with Eq. (23) (see Pottmann and Wallner [7] and Sommerville [5]). Coordinate x_0 has been set to unity in order to find the distance between the quadric and line $\mathcal{S}$ in Euclidean space.

$$\begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{23}$$

Therefore, the radial line $\mathcal{N}_r$, defined by point X and the normal to the quadric on this point, has the following Plücker line coordinates: $\mathcal{N}_r\{N_1 : N_2 : N_3 : x_2N_3 - x_3N_2 : x_3N_1 - x_1N_3 : x_1N_2 - x_2N_1\}$. In order to find the shortest distance, line $\mathcal{N}_r$ has to be perpendicular to line $\mathcal{S}$. This condition appears as Eq. (24) which is simply the dot product of the two direction vectors.

$$N_1S_{23} + N_2S_{31} + N_3S_{12} = 0 \tag{24}$$

In Eq. (24), N_1, N_2 and N_3 are linear function of x_1, x_2 and x_3. It is therefore easy to obtain the expression, Eq. (25), relating one variable to the two others.

$$\begin{aligned}(s_{01}a_{11} + s_{02}a_{12} + s_{03}a_{13})x_1 - (s_{01}a_{12} + s_{02}a_{22} + s_{03}a_{23})x_2 \\ +(a_{13}s_{01} + a_{23}s_{02} + a_{33}s_{03})x_3 - (s_{01}a_{01} + s_{02}a_{02} + s_{03}a_{03}) = 0\end{aligned} \tag{25}$$

Line $\mathcal{N}_r$ also has to intersect line $\mathcal{S}$ which can be written as Eq. (26). Substituting Eq. (25) in Eq. (26) to eliminate x_3, a second order bivariate in terms of x_1 and x_2 is obtained.

$$\begin{aligned}N_1s_{23} + N_2s_{31} + N_3s_{12} + (x_2N_3 - x_3N_2)s_{01} \\ +(x_3N_1 - x_1N_3)s_{02} + (x_1N_2 - x_2N_1)s_{03} = 0\end{aligned} \tag{26}$$

Also, recall that point X is on the quadric surface therefore it has to satisfy Eq. (15). Again, substituting Eq. (25) in Eq. (15) returns another second order bivariate in terms of x_1 and x_2. The two bivariates obtained with Eq. (25) and Eq. (26) can be rewritten in the form of Eqs. (27,28) respectively. These coefficients are too long to write.

$$\theta_1 x_2^2 + (\alpha_1 x_1 + \gamma_1)x_2 + (\omega_1 x_1^2 + \sigma_1 x_1 + \eta_1) = 0 \quad (27)$$

$$\theta_2 x_2^2 + (\alpha_2 x_1 + \gamma_2)x_2 + (\omega_2 x_1^2 + \sigma_2 x_1 + \eta_2) = 0 \quad (28)$$

A fourth order univariate can be obtained using Bezout's method as described by Salmon [8]. The univariate obtained by eliminating x_2 is given in Eq. (29).

$$\begin{aligned}
&(-\theta_2\alpha_1^2\omega_2 + \theta_1\alpha_2\alpha_1\omega_2 - \theta_1^2\omega_2^2 - \theta_1\alpha_2^2\omega_1 + \theta_2\alpha_1\alpha_2\omega_1 - \theta_2^2\omega_1^2 + 2\theta_1\omega_2\theta_2\omega_1)x_1^4 \\
&+(-\theta_1\alpha_2^2\sigma_1 - 2\theta_1^2\omega_2\sigma_2 + \theta_1\alpha_2\alpha_1\sigma_2 + \theta_1\alpha_2\gamma_1\omega_2 - \theta_2\alpha_1^2\sigma_2 + \theta_1\gamma_2\alpha_1\omega_2 \\
&+\theta_2\alpha_1\alpha_2\sigma_1 - 2\theta_1\alpha_2\gamma_2\omega_1 + 2\theta_1\omega_2\theta_2\sigma_1 - 2\theta_2\alpha_1\gamma_1\omega_2 + 2\theta_1\sigma_2\theta_2\omega_1 \\
&-2\theta_2^2\omega_1\sigma_1 + \theta_2\alpha_1\gamma_2\omega_1 + \theta_2\gamma_1\alpha_2\omega_1)x_1^3 + (-2\theta_1^2\omega_2\eta_2 - \theta_1\gamma_2^2\omega_1 - \theta_1^2\sigma_2^2 \\
&-\theta_2\alpha_1^2\eta_2 + \theta_1\alpha_2\alpha_1\eta_2 + \theta_2\gamma_1\gamma_2\omega_1 + 2\theta_1\sigma_2\theta_2\sigma_1 + \theta_2\alpha_1\alpha_2\eta_1 - \theta_1\alpha_2^2\eta_1 \\
&+\theta_1\alpha_2\gamma_1\sigma_2 + 2\theta_1\omega_2\theta_2\eta_1 + \theta_1\gamma_2\alpha_1\sigma_2 - 2\theta_2^2\omega_1\eta_1 + \theta_1\gamma_2\gamma_1\omega_2 + \theta_2\alpha_1\gamma_2\sigma_1 \\
&+2\theta_1\eta_2\theta_2\omega_1 - 2\theta_2\alpha_1\gamma_1\sigma_2 - 2\theta_1\alpha_2\gamma_2\sigma_1 - \theta_2\gamma_1^2\omega_2 - \theta_2^2\sigma_1^2 + \theta_2\gamma_1\alpha_2\sigma_1)x_1^2 \\
&+(\theta_2\gamma_1\gamma_2\sigma_1 + \theta_2\alpha_1\gamma_2\eta_1 - 2\theta_2^2\sigma_1\eta_1 - 2\theta_2\alpha_1\gamma_1\eta_2 - 2\theta_1^2\sigma_2\eta_2 - \theta_2\gamma_1^2\sigma_2 \\
&+\theta_1\gamma_2\gamma_1\sigma_2 + \theta_1\alpha_2\gamma_1\eta_2 - \theta_1\gamma_2^2\sigma_1 + 2\theta_1\eta_2\theta_2\sigma_1 - 2\theta_1\alpha_2\gamma_2\eta_1 + \theta_2\gamma_1\alpha_2\eta_1 \\
&+2\theta_1\sigma_2\theta_2\eta_1 + \theta_1\gamma_2\alpha_1\eta_2)x_1 + \theta_2\gamma_1\gamma_2\eta_1 + \theta_1\gamma_2\gamma_1\eta_2 - \theta_1\gamma_2^2\eta_1 - \theta_2\gamma_1^2\eta_2 \\
&+2\theta_1\eta_2\theta_2\eta_1 - \theta_1^2\eta_2^2 - \theta_2^2\eta_1^2 = 0 \qquad (29)
\end{aligned}$$

Solving Eq. (29) returns four values of x_1. The corresponding values of x_2 can be calculated with Eq. (30) that has been obtained by eliminating x_2^2 from Eqs. (27,28). Then x_3 can be easily calculated using Eq. (25).

$$x_2 = \frac{(\theta_1\omega_2 - \theta_2\omega_1)x_1^2 + (\theta_1\sigma_2 - \theta_2\sigma_1)x_1 + \theta_1\eta_2 - \theta_2\eta_1}{(\theta_2\alpha_1 - \theta_1\alpha_2)x_1 + \theta_2\alpha_1 - \theta_1\alpha_2} \quad (30)$$

To each point X on the quadric, another point Y on line S has to be found to calculate the desired distance. Point Y is the intersection of line $\mathcal{N}_r\{N_1 : N_2 : N_3 : x_2N_3 - x_3N_2 : x_3N_1 - x_1N_3 : x_1N_2 - x_2N_1\}$ and line $\mathcal{S}$. Using the equation of a point on the line (two from line $\mathcal{N}_r$ and one from line $\mathcal{S}$) the system of Eq. (31) can be easily obtained. The four pairs of points X and Y may be compared to select the shortest distance between the quadric and line $\mathcal{S}$. There are other ways to find this distance and two will be discussed in Section 5 however the one described here is complete in the sense that it finds the distance as well as the coordinates of the closest points on the quadric and the line.

$$\begin{bmatrix} N_{01} & N_{02} & N_{03} \\ 0 & N_{12} & -N_{31} \\ -S_{12} & 0 & S_{23} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ N_{01} \\ S_{02} \end{bmatrix} \quad (31)$$

4.2 Application of the Distance as a Metric

A configuration with no real transversal can be obtained by changing the orientation of line $\mathcal{P}_a$ in the system shown in Figure 1 and Table 1. $\mathcal{P}_a$ used to point in the z-axis direction. Now $\mathcal{P}'_a\{-3 : 0 : 1 : 0 : 1 : 0\}$ points in the y-axis direction. Figure 2(d) shows that it will no longer intersect the quadric. The four possible quadrics are shown in Figure 3. This condition of non-intersection is evident in each case and the fourth, external line is connected to the quadric with a normal line segment that clearly indicates the shortest distance. These four lengths have been calculated and appear in Table 2.

Table 2: Shortest distance between a line and a quadric surface.

Quadric	Distance
$\mathcal{P}'\mathcal{QR}$	0.5495
$\mathcal{P}'\mathcal{QS}$	1.5019
$\mathcal{P}'\mathcal{RS}$	0.5920
$\mathcal{QRS}$	0.8059

Table 2 shows that, given four lines, the shortest distance between a line and the quadric surface ruled by the three others is not the same for the four possible quadrics that can be generated. Consequently, this quantity cannot, without ambiguity, serve as a measure of proximity to singularity in general 4R positioning manipulators.

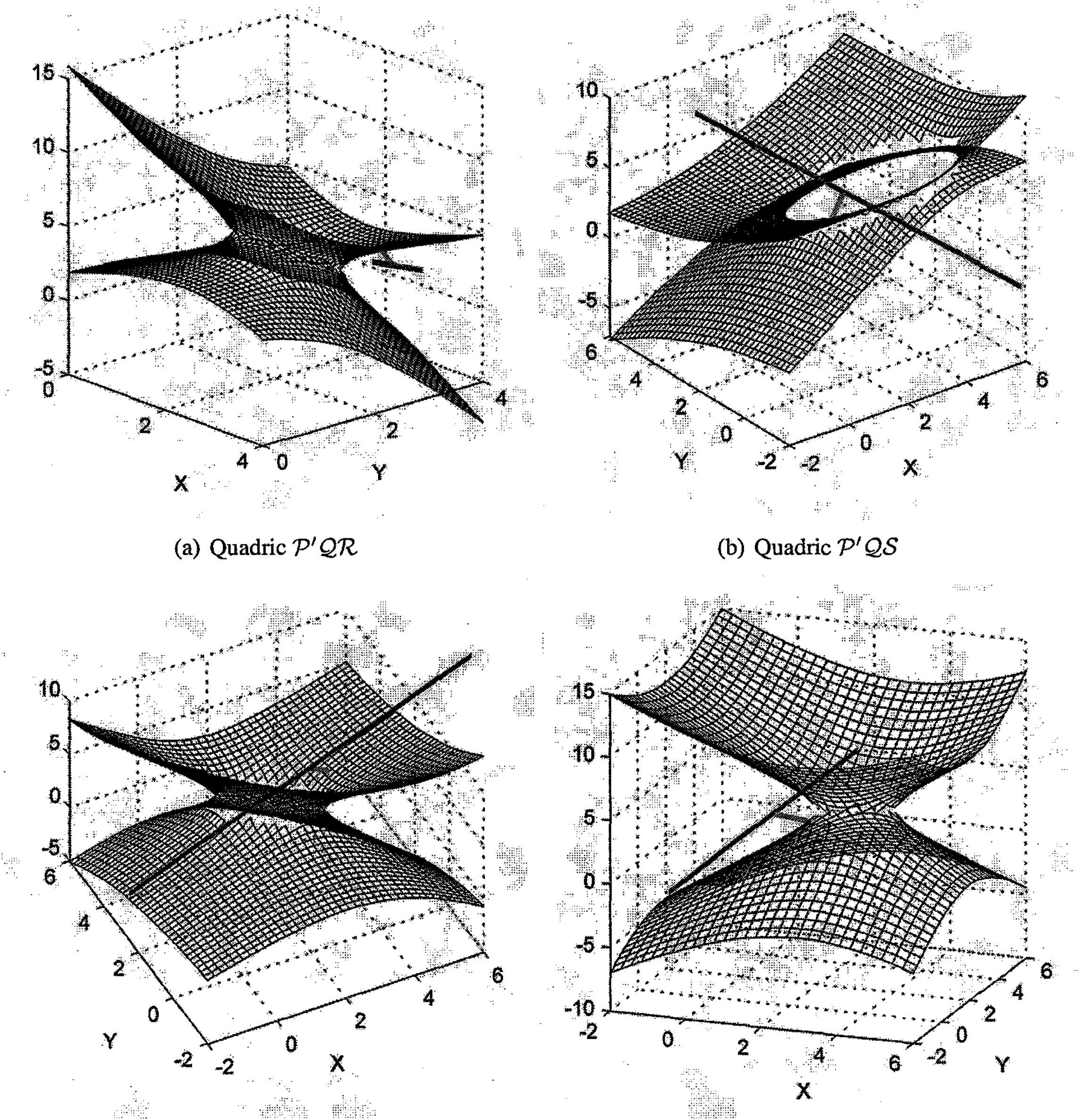
5 USEFUL GEOMETRIC IMPLICATIONS

Although the shortest distance between line and quadric is not a useful singularity proximity metric, finding it provides an opportunity to introduce some interesting, even useful, geometry.

- This problem is simplified by reducing it to the two dimensional one of finding the shortest distance between a point and a conic.
- The point-to-conic distance problem illustrates the advantages gained by resorting to so-called symmetrically ideal formulation of constraint equations.
- The point to conic distance is a key element in fitting digital image pixels to expected shapes in automated, camera aided metrology as described in O'Leary and Zsombor-Murray [9].
- This problem formulation can be simplified to a still greater extent by using a simple application of discriminant theory, a precis of which appears in Akritas [10], to find double roots of higher order univariate polynomial equations.

5.1 Quadric, Tangent Line and Tangent Plane

Examine Fig. 4. Shown there is a given quadric a, represented by a general hyperboloid of one sheet, and a given line $\mathcal{G}_r\{1 : 0 : 0 : 0 : 0 : 0\}$. There is no loss in generality if the x_1-axis is chosen to be the given line. One seeks the point X on surface a that is closest to line $\mathcal{G}$.



(a) Quadric $\mathcal{P}'\mathcal{QR}$

(b) Quadric $\mathcal{P}'\mathcal{QS}$

(c) Quadric $\mathcal{P}'\mathcal{RS}$

(d) Quadric $\mathcal{QRS}$

Figure 3: Four possible quadrics; their remaining line and the shortest distance.

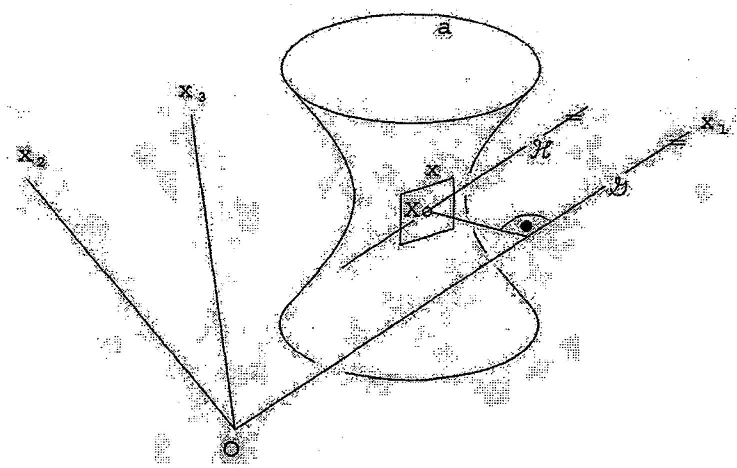


Figure 4: Line and Quadric

This is a two stage process. The first stage involves reducing this spatial problem to a planar one involving extreme distances from point to conic. Then one may invoke the "four concentric tangent circles, centred on the point, to the conic" model that admits a quartic solution.

5.1.1 Constraint Equations in 3D

In addition to x, the tangent plane to point X on a, a line $\mathcal{H}$ parallel to $\mathcal{G}$ is introduced. It is observed that

$$\mathcal{H} \parallel \mathcal{G}, \;\; \mathcal{H} \in x, \;\; X \in \mathcal{H}, \;\; X \in a$$

so one may immediately write the Plücker coordinates of $\mathcal{H}$ and polar tangency relation for x with respect to a, *i.e.*, $X \in x$. Solutions in Euclidean space allow setting $x_0 = 1$.

$$\mathcal{H}_r\{1:0:0:0:h_{31}:h_{12}\} \equiv \mathcal{H}_a\{0:H_{02}:H_{03}:1:0:0\}$$

$$x \equiv \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} a_{00} + a_{01}x_1 + a_{02}x_2 + a_{03}x_3 \\ a_{01} + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{02} + a_{12}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{03} + a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \end{bmatrix} \tag{32}$$

Now the conditions $\mathcal{H} \in x$ and $X \in \mathcal{H}$ are expressed with the conventional relations that declare, respectively, the nonexistence of the intersection of line and plane to denote containment and the nonexistence of a plane spanned by line and point to denote point-on-line. This will relate X_i to h_{ij} in the first instance and x_i to H_{ij} in the second.

$$\begin{array}{ll} h_{01}X_1 + h_{02}X_2 + h_{03}X_3 = 0 & \rightarrow X_1 = 0 \\ -h_{01}X_0 + h_{12}X_2 - h_{31}X_3 = 0 & \rightarrow X_0 - h_{12}X_2 + h_{31}X_3 = 0 \\ -h_{02}X_0 - h_{12}X_1 + h_{23}X_3 = 0 & \\ -h_{03}X_0 + h_{31}X_1 - h_{23}X_2 = 0 & \end{array} \tag{33}$$

$$
\begin{aligned}
&H_{01}x_1 + H_{02}x_2 + H_{03}x_3 = 0 \\
&-H_{01}x_0 + H_{12}x_2 - H_{31}x_3 = 0 \\
&-H_{02}x_0 - H_{12}x_1 + H_{23}x_3 = 0 \quad \rightarrow x_3 - h_{31} = 0 \\
&-H_{03}x_0 + H_{31}x_1 - H_{23}x_2 = 0 \quad \rightarrow x_2 + h_{12} = 0
\end{aligned} \tag{34}
$$

Combining the first result from Eq. (33) with the second row of Eq. (32) produces a constraint, Eq. (35), on x_i in terms of conic coefficients a_{ij}.

$$a_{01} + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \tag{35}$$

Then both results from Eq. (34) are used to eliminate h_{ij} from the second result in Eq. (33) to produce $X_0 + X_2x_2 + X_3x_3 = 0$ which can be reformulated in terms of quadric coefficients a_{ij} and point variables x_i by using the remaining three lines in Eq. (32). The second constraint is Eq. (36).

$$(a_{00}+a_{01}x_1+a_{02}x_2+a_{03}x_3)+(a_{02}+a_{12}x_1+a_{22}x_2+a_{23}x_3)x_2+(a_{02}+a_{12}x_1+a_{23}x_2+a_{33}x_3)x_3 = 0 \tag{36}$$

Eliminating x_1 between Eq. (35) and Eq. (36) gives a conic in x_2 and x_3, *i.e.*, Eq. (37).

$$
\begin{aligned}
&(a_{00}a_{11} - a_{01}^2) + 2(a_{02}a_{11} - a_{01}a_{12})x_2 + 2(a_{03}a_{11} - a_{01}a_{13})x_3 \\
&+(a_{11}a_{22} - a_{12}^2)x_2^2 + 2(a_{11}a_{23} - a_{12}a_{13})x_2x_3 + (a_{11}a_{33} - a_{13}^2)x_3^2 = 0
\end{aligned} \tag{37}
$$

This conic, really a cylinder parallel to x_1-axis, replaces a in the exercise where the extreme distances from a point to a conic were derived, thus beginning stage two. If the quadric point equation, Eq. (15), is used to eliminate x_1, instead of Eq. (36), the resulting bivariate is identical to Eq. (37). One cannot obtain a univariate at this stage. Only by eliminating x_1, however, will a point projection of lines $\mathcal{G}$ and $\mathcal{H}$ appear together with the conic Eq. (37). The variables x_2 and x_3 that remain here become x_1 and x_2, respectively, in the next, planar stage.

5.2 A Conic and Four Tangent Circles

Examine Fig. 5. Shown there is a given conic a, represented by a general ellipse, and a given coplanar point $P\{p_0 : p_1 : p_2\}$. The problem is to find the point $X\{x_0 : x_1 : x_2\}$ on a that is closest to P.

An equivalent problem, the one chosen to be solved, is to find lines $x\{X_0 : X_1 : X_2\}$ that are cotangential to a and a circle centred on P and on X the point of tangency. Clearly there are four of these. The one shown on X near P is the closest. Another, not shown, is *farthest* from P. It is associated with the tangent circle of greatest radius and touches on the opposite side of a. There are two more tangent circles, concentric on P, shown in this example indicating that this problem configuration admits four real solutions

5.2.1 Constraint Equations in 2D

One seeks a solution in the Euclidean plane with $x_0 = 1$. There is no loss in generality if P is on the origin and the scalar equation, expressing $X \in k$, where k is the circle centred on P, is Eq. (38).

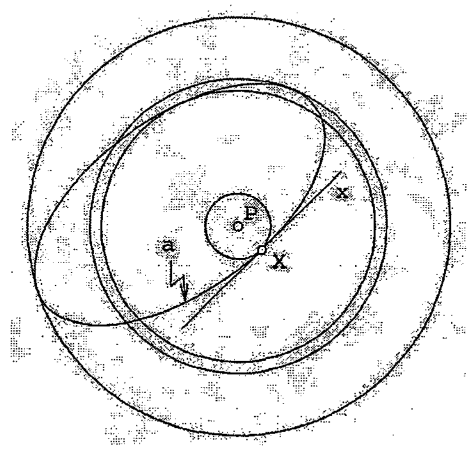


Figure 5: Point and Conic

$$[1\ x_1\ x_2]\begin{bmatrix}-r^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ x_1\\ x_2\end{bmatrix} = x_1^2 + x_2^2 - r^2 = 0 \tag{38}$$

The second constraint equation, Eq. (39), defines the coefficients or homogeneous coordinates of the polar line x tangent to a on X.

$$\begin{bmatrix}a_{00} & a_{01} & a_{02}\\ a_{01} & a_{11} & a_{12}\\ a_{02} & a_{12} & a_{22}\end{bmatrix}\begin{bmatrix}1\\ x_1\\ x_2\end{bmatrix} = \begin{bmatrix}a_{00}+a_{01}x_1+a_{02}x_2\\ a_{01}+a_{11}x_1+a_{12}x_2\\ a_{02}+a_{12}x_1+a_{22}x_2\end{bmatrix} = \begin{bmatrix}X_0\\ X_1\\ X_2\end{bmatrix} = \lambda\begin{bmatrix}-r^2\\ x_1\\ x_2\end{bmatrix} \tag{39}$$

The last vector states that x is tangent on k and the three rows of Eq. (39) provide, together with Eq. (38), the four constraints necessary to handle the four variables $x_1,\ x_2,\ r^2,\ \lambda$. First λ is eliminated among the equations expressed by the three rows of Eq. (39). Then r^2 is eliminated among these two resultants and Eq. (38) to yield the two cubics Eq. (40).

$$\begin{aligned}a_{00}x_0^2x_1 + 2a_{01}x_0x_1^2 + a_{02}x_0x_1x_2 + a_{01}x_0x_2^2 + a_{11}x_1^3 + a_{12}x_1^2x_2 + a_{11}x_1x_2^2 + a_{12}x_2^3 = 0\\ a_{00}x_0^2x_2 + a_{02}x_0x_1^2 + a_{01}x_0x_1x_2 + 2a_{02}x_0x_2^2 + a_{12}x_1^3 + a_{22}x_1^2x_2 + a_{12}x_1x_2^2 + a_{22}x_2^3 = 0\end{aligned} \tag{40}$$

The variable $x_0 = 1$ is included to present homogeneous cubics with terms ordered lexographically.

5.2.2 A Univariate Quartic

The resultant of Eq. (40), Eq. (41), is a univariate polynomial of degree seven in x_1, a product of linear, quadratic and quartic factors. The linear and quadratic factors represent degenerate solutions. The geometric significance of these has not been determined but it is clear they cannot represent complete, valid solutions because the linear factor is devoid of any conic coefficients while the quadratic factor lacks $a_{11},\ a_{12}$ and a_{22}.

$$x_1(Ax_1^2 + Bx_1 + C)(Dx_1^4 + Ex_1^3 + Fx_1^2 + Gx_1 + H) = 0 \qquad (41)$$

5.2.3 The Coefficients

The eight coefficients in Eq. (41) are tabulated below in Eq. (42) to show the computational effort required to obtain them. When solving such a problem one would evaluate the quartic. The coefficients A, B and C indicate that the quadratic can generate only complex values of x_1.

$$\begin{aligned}
A = a_{01}^2 + a_{02}^2,\quad B = 2a_{00}a_{01},\quad C = a_{00}^2 \\
D = (a_{11}a_{22} - a_{12}^2)(a_{11}^2 + a_{22}^2 + 4a_{12}^2 - 2a_{11}a_{22}) \\
E = (4a_{22}a_{11}^2 - 2a_{12}^2a_{11} + 10a_{22}a_{12}^2 + 2a_{22}^3 - 6a_{22}^2a_{11})a_{01} \\
-2a_{02}a_{12}a_{11}^2 - 12a_{02}a_{12}^3 - 2a_{02}a_{22}^2a_{12} + 8a_{02}a_{22}a_{11}a_{12} \\
F = (a_{22}^3 + 4a_{22}a_{12}^2 - 2a_{22}^2a_{11} + a_{22}a_{11}^2)a_{00} \\
-12a_{02}^2a_{12}^2 + 2a_{02}^2a_{22}a_{11} - a_{02}^2a_{22}^2 - 6a_{02}a_{12}a_{11}a_{01} \\
+12a_{02}a_{22}a_{01}a_{12} + 5a_{22}a_{01}^2a_{11} - 4a_{22}^2a_{01}^2 \\
G = (-2a_{02}a_{12}a_{11} + 4a_{02}a_{12}a_{22} - 2a_{22}^2a_{01} + 2a_{12}^2a_{01} + 2a_{22}a_{01}a_{11})a_{00} \\
-4a_{12}a_{02}^3 + 2a_{22}a_{01}a_{02}^2 - 4a_{02}a_{12}a_{01}^2 + 2a_{22}a_{01}^3 \\
H = (a_{01}^2a_{22} - 2a_{01}a_{02}a_{12} + a_{00}a_{12}^2)a_{00}
\end{aligned} \qquad (42)$$

5.2.4 A Better Frame of Formulation

Redoing this problem with $a_{01} = a_{02} = a_{12} = 0$ and $P \neq P(0,0)$ leads to a more compact solution. A quintic, without the degenerate quadratic factor found in Eq. (41) and with much simpler coefficients, is produced directly. Using the standard form conic scalar equation coefficient matrix and that of a circle centred on $P(p_1, p_2)$, for the common polar line relation, and the scalar equation for points on a circle,

$$\begin{bmatrix} a_{00} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{bmatrix}, \quad \begin{bmatrix} p_1^2 + p_2^2 - r^2 & -p_1 & -p_2 \\ -p_1 & 1 & 0 \\ -p_2 & 0 & 1 \end{bmatrix}, \quad (x_1 - p_1)^2 + (x_2 - p_2)^2 - r^2 = 0$$

yields the following quartic univariate in x_1 after removing the linear factor $(x_1 - p_1)$.

$$Px_1^4 + Qx_1^3 + Rx_1^2 + Sx_1 + T = 0$$

where

$$\begin{aligned}
P = a_{11}(a_{11} - a_{22})^2,\quad Q = 2a_{11}a_{22}p_1(a_{11} - a_{22}) \\
R = a_{11}a_{22}(a_{11}p_2^2 - 2a_{00} + a_{22}p_1^2) + a_{00}(a_{11}^2 + a_{22}^2) \\
S = 2a_{00}a_{22}p_1(a_{11} - a_{22}),\quad T = a_{00}a_{22}^2p_1^2
\end{aligned}$$

5.3 Distance from Line and Point to Conic via Discriminants

Essentially this technique yields the same results as does the procedure using conditions of tangent line polarity. The advantage is that the intermediate steps in symbolic algebra are much simpler and easy to follow.

5.3.1 Line to Conic

This procedure involves writing the scalar equations of the line and conic and solving for the variable c but using the given coefficients a and b below. Determining c will produce parallel lines to the one given and tangent to the conic.

$$ax_1 + bx_2 + c = 0, \quad a_{00} + 2a_{01}x_1 + 2a_{02}x_2 + a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = 0$$

The resultant quadratic in x_1 is

$$(a^2a_{22} - 2aba_{12} + b^2a_{11})x_1^2 - 2(aba_{02} + bca_{12} - caa_{22} - b^2a_{01})x_1 + b^2a_{00} - 2bca_{02} + c^2a_{22} = 0$$

Taking the first derivative with respect to x_1 yields a linear equation which, when solved simultaneously with the quadratic to eliminate x_1, produces a quadratic in c. With these, two tangent lines equations can be formulated and their polarity with respect to the conic will produce the necessary points from which to drop normals to the given line. The discriminant of a quadratic is well known as the expression within the square root of the quadratic formula but this simple problem using the derivative is used to introduce the condition of double roots of a univariate of higher degree.

5.3.2 Point to Conic

Recall the four concentric circles shown in Fig. 5. The tangency between circle and conic is what is desired and one begins by intersecting a standard form conic with a circle of unspecified radius centred on the given point. These equations are written below.

$$a_{00} + a_{11}x_1^2 + a_{22}x_2^2 = 0, \quad p_1^2 + p_2^2 - r^2 - 2p_1x_1 - 2p_2x_2 + x_1^2 + x_2^2 = 0$$

Eliminating x_2 produces a univariate of degree four in x_1. Taking the first derivative with respect to x_1 produces a cubic in x_1. Eliminating x_1 from these two equations yields a sextic in r^2 where the r are the radii of the circles, the least having the magnitude of the smallest distance from the point to the conic.

6 CONCLUSION

A geometric approach was used to reveal singularities of 4R positioning manipulators. All points on the EE that are also on a line that intersects all R-joint axes will not be able, instantaneously, to move along this line. In the case of a 4R manipulator, there are two possible real transversals and a way to find them was presented. It was also shown that, for four given lines, if any of them is respectively tangent to or intersects the quadric ruled by the three others, a double or two real transversal(s) exist. On the other hand, if the line does not intersect the quadric, there is no real transversal and it was shown with a numerical example that the shortest distance between the two geometric elements cannot be immediately used as a metric to indicate proximity to a singular pose. Nevertheless, an in-depth study of the problem of finding the shortest distance between a line and a quadric gave us the opportunity to introduce interesting and useful geometry techniques like reducing the problem to a 2D formulation and solving it using the discriminant theory. Finally, in an extension to this work, one might pursue further the notion of calculating a meaningful shortest distance for the metric. For example, it is easy to compute shortest distances to the actual R-joint axes and study how the manipulator may come to acquire a pose that entails lines of singular direction.

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