CONJUGATION IN THE DISPLACEMENT GROUP AND MOBILITY IN MECHANISMS

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ABSTRACT
The paper deals with the Lie group algebraic structure of the set of Euclidean displacements, which represent rigid-body motions. We begin by looking for a representation of a displacement, which is independent of the choice of a frame of reference. Then, it is a simple matter to prove that displacement subgroups may be invariant by conjugation. This mathematical tool is suitable for solving special problems of mobility in mechanisms.

CONJUGAISON DANS LE GROUPE DES DÉPLACEMENTS ET MOBILITÉ DANS LES MÉCANISMES

RÉSUMÉ
L'article traite de la structure algébrique de groupe de Lie de l'ensemble des déplacements euclidiens, lesquels représentent les mouvements des corps rigides. Nous commençons par chercher une représentation d'un déplacement qui est indépendante du choix d'un repère de référence. Alors, il est simple de prouver que des sous-groupes de déplacements peuvent être invariants par conjugaison. Cet outil mathématique est adapté à la résolution de problèmes spéciaux de mobilité dans des mécanismes.

1. PREAMBLE

In standard Euclidean geometry, the set of the points has the algebraic properties of an Euclidean affine 3-dimensional (3D) space. The elements of this set, i.e., the points are denoted by capital letters, for example M, M', N, C… An ordered pair of points determines a *bound* vector. A member of the equivalence class of bound vectors, which are equipollent to a given bound vector is called a *free* vector. The set of free vectors is the Euclidean vector 3D space. Henceforth, in what follows, vector means free vector. Vectors obtained from two ordered points will be denoted by a sequence of two bold-faced capital letters, for example, (**MM'**). An alternative notation of (**MM'**) is M' – M. A vector may be given independently of points, in which case a single bold-faced character, for example **V**, is employed. Small bold-faced letters, for example **u**, **v**, **w**, **i**, **j**, **k** designate unit vectors.

Classical operations between vectors are the scalar product (or dot product), and the vector product denoted herein by $\wedge$. The cross $\times$ will be used for the Cartesian product of two sets and for the direct product of two subgroups of the group of displacements.

Affine subspaces of the 3D affine space are affine planes and affine lines. Vector subspaces of the 3D vector space are called vector planes and vector lines. A vector plane is also called a plane direction and a vector line, a line direction. An affine line, which may be an axis of rotation, is determined and designated by a frame of reference of an 1D affine space, i.e., by a point and a unit vector, for instance (N, **w**).

An arrow is used to indicate a point transformation; M *becomes* M' is written M $\rightarrow$ M'. Operators are used for denoting point transformations. For example, we write M' = D M, where D is an operator acting on any point written at its right side. If M $\rightarrow$ M'= D M and M' $\rightarrow$ M"= D' M', then we have M $\rightarrow$ M"= $D'D$ M. The transformation $D'D$ is the composition product of the transformations D and D'.

This preamble is justified by the fact that several authors do not clearly discriminate point and vector, which are considered to be elements of $\mathbb{R}^3$.

2. EXPRESSION OF A DISPLACEMENT

In a given orthonormal frame of reference R = (O, **i**, **j**, **k**), which includes a point O called origin and a vector base (**i**, **j**, **k**), a rigid-body motion or displacement D is represented by a matrix operation $D_{/\mathrm{R}}$, which belongs to the group usually denoted $SE(3)$ in the theory of matrix Lie groups. A noteworthy work based on matrix Lie subgroups of $SE(3)$ was published recently in [1].

If the coordinates (x, y, z) of any point M make up a (3×1) column matrix $\mathrm{M}_{/\mathrm{R}}$, then the displaced point M', which is represented by the matrix $\mathrm{M'}_{/\mathrm{R}}$ is

$$\mathrm{M}_{/\mathrm{R}} \rightarrow \mathrm{M'}_{/\mathrm{R}} = A_{/\mathrm{R}} \mathrm{M}_{/\mathrm{R}} + \mathrm{T}_{/\mathrm{R}} \tag{1}$$

where $A_{/\mathrm{R}}$ is a (3×3) matrix belonging to the matrix group usually denoted $SO(3)$ and $T_{/\mathrm{R}}$ is a (3×1) matrix.

We use the symbolic notation:

$$\begin{aligned} \mathrm{M'}_{/\mathrm{R}} &= D_{/\mathrm{R}} \mathrm{M}_{/\mathrm{R}} \\ D_{/\mathrm{R}} &= (A_{/\mathrm{R}}, T_{/\mathrm{R}}) \end{aligned} \tag{2}$$

The operator $D_{/\mathrm{R}}$ represents a displacement referred to the frame R. It is clear that the notation can be simplified: M' = A M + T = D M. However, all mathematical entities depend on the chosen frame. The following formulas concern the product of elements belonging to the set $\mathcal{D}_{/R}$ of the operators $D_{/R}$. It is straightforward to verify that the product of displacements is the binary operation of an algebraic group. Furthermore, the group has also the algebraic properties of a smooth 6D manifold and is a Lie group.

$$\begin{aligned} & D_1 = (A_1, T_1) \text{ and } D_2 = (A_2, T_2) \Rightarrow D_2 D_1 = (A_2 A_1, T_2 + A_2 T_1) \\ & D^{-1} = (A^{-1}, -A^{-1} T) \\ & D_1 D D_1{}^{-1} = (A_1 A A_1{}^{-1}, T_1 + A_1 T - A_1 A A_1{}^{-1} T_1) = C_1 \end{aligned} \tag{3}$$

The displacement C_1 is the *conjugate* of the displacement D by the displacement D_1. The conjugation by D_1 is also valid when D is replaced by any subset $\mathcal{B}$ of displacements. The set of the conjugates by D_1 of the elements of $\mathcal{B}$ is the subset $\mathcal{C}_1 = D_1 \mathcal{B} D_1^{-1}$. The conjugation between $\mathcal{B}$ and $\mathcal{C}_1$ is a relation of equivalence as defined in abstract algebra. All the equivalent conjugates of $\mathcal{B}$ constitute the conjugacy class of $\mathcal{B}$. A type of displacement set is characterized by the conjugacy class of a specimen of the type. For example, a set of rotations about a given axis designates a type of displacements; a set of rotations about another axis is a conjugate of the first set by a displacement, which transforms the first axis into the second axis. The conjugacy class is made of all the rotation sets about all the axes of the space.

Points and vectors can be defined without using a frame of reference; else, one could not define the frame of reference. It is convenient to express a displacement by means of points and vectors [2]. The first step in the intrinsic formulation of a displacement is the frame-free expression of the orthogonal matrix A. A represents a rotation, which is characterized by a vector $\theta\,\mathbf{r}$, where θ is the angle of rotation and $\mathbf{r}$ is a unit vector parallel to the axis of rotation. Using the formula about the double vector product, one can verify $\mathbf{r}\wedge[\mathbf{r}\wedge(\mathbf{r}\wedge \mathbf{V})] = -\,\mathbf{r}\wedge \mathbf{V}$ for any vector $\mathbf{V}$. Let $\mathbf{r}^\wedge$ denote the skew-symmetric linear operator of the vector product by $\mathbf{r}$; the foregoing property is also expressed by: $(\mathbf{r}^\wedge)^3 = -\,\mathbf{r}^\wedge$.

The notation $\exp(\theta\,\mathbf{r}^\wedge)\,\mathbf{V}$ designates the exponential series of the linear operator $\theta\,\mathbf{r}^\wedge$ acting on the vector $\mathbf{V}$.

$$\begin{aligned} \exp(\theta\,\mathbf{r}^\wedge)\mathbf{V} &= \mathbf{V} + (\theta/1!)\,\mathbf{r}\wedge\mathbf{V} + (\theta^2/2!)\,\mathbf{r}\wedge(\mathbf{r}\wedge\mathbf{V}) + \ldots + (\theta^n/n!)\,(\mathbf{r}^\wedge)^n\mathbf{V} + \ldots \\ &= \mathbf{V} + \sin\theta\mathbf{r}\wedge\mathbf{V} + (1-\cos\theta)\mathbf{r}\wedge(\mathbf{r}\wedge\mathbf{V}) \\ &= [1 + \sin\theta\mathbf{r}^\wedge + (1-\cos\theta)(\mathbf{r}^\wedge)^2]\mathbf{V} \end{aligned} \tag{4}$$

This is the intrinsic vector formulation of the Rodrigues formula for rotation. The exponential is a typical feature of Lie's theory of continuous groups.

Replacing the (3×3) matrix A by $\exp(\theta\,\mathbf{r}^\wedge)$ and the (3×1) matrix T by the vector $\mathbf{T}$, we obtain a coordinate-free formulation of the displacement for any point M, namely,

$$(\mathbf{OM}) \rightarrow (\mathbf{OM}') = \mathbf{T} + \exp(\theta\,\mathbf{r}^\wedge)(\mathbf{OM}) \tag{5}$$

or

$$\mathrm{M}' = \mathrm{O} + \mathbf{T} + \exp(\theta\,\mathbf{r}^\wedge)(\mathbf{OM}) \tag{6}$$

However, the foregoing two expressions depend on the choice of the origin O. Mozzi-Chasles' theorem states that, except when $\mathbf{r}$ is the zero vector, any displacement is a screw motion with an axis. Let H be the foot of the perpendicular drawn from O to the sought axis. The vector (**OH**) can be calculated [3]:

$$\begin{aligned}(\mathbf{OH}) &= [-\sin(\theta/2)\mathbf{r}\wedge(\mathbf{r}\wedge\mathbf{T})+\mathbf{cos}(\theta/2)\mathbf{r}\wedge\mathbf{T}]/[2\sin(\theta/2)] \\ &= [2\sin(\theta/2)]^{-1}\{\exp[-(\theta/2)\mathbf{r}^\wedge](\mathbf{r}\wedge\mathbf{T})\}\end{aligned} \tag{7}$$

The points C of the screw axis are determined by the parametric representation:

$$(\mathbf{OC}) = (\mathbf{OH}) + h\,\mathbf{r} \text{ or } \mathrm{C} = \mathrm{H} + h\,\mathbf{r}. \tag{8}$$

where h is a real parameter called abscissa of C in the frame (H, **r**).

An origin-free expression of the displacement will be derived.

$$\begin{aligned}\mathrm{M}' &= \mathrm{O} + \mathbf{T} + \exp(\theta\,\mathbf{r}^\wedge)[(\mathbf{OH})+(\mathbf{HC})+(\mathbf{CM})] \\ &= \mathrm{O} + \mathbf{T} + \exp(\theta\,\mathbf{r}^\wedge)(\mathbf{OH}) + \exp(\theta\,\mathbf{r}^\wedge)(\mathbf{HC}) + \exp(\theta\,\mathbf{r}^\wedge)(\mathbf{CM})\end{aligned} \tag{9}$$

exp(θ **r**^)(**HC**) is equal to (**HC**) because (**HC**) is parallel to **r**.

$$\begin{aligned}\exp(\theta\,\mathbf{r}^\wedge)(\mathbf{OH}) &= [2\sin(\theta/2)]^{-2}\exp(\theta\,\mathbf{r}^\wedge)\exp[(-\theta/2)\mathbf{r}^\wedge](\mathbf{r}\wedge\mathbf{T}) \\ &= \mathbf{r}\wedge(\mathbf{r}\wedge\mathbf{T}) + (\mathbf{OH})\end{aligned} \tag{10}$$

$$\begin{aligned}\mathrm{M}' &= \mathrm{C} + (\mathbf{OH}) + (\mathbf{HC}) + \mathbf{T} + \mathbf{r}\wedge(\mathbf{r}\wedge\mathbf{T}) + \exp(\theta\,\mathbf{r}^\wedge)(\mathbf{CM}) \\ \mathrm{M}' &= \mathrm{C} + \mathbf{T} + \mathbf{r}\wedge(\mathbf{r}\wedge\mathbf{T}) + \exp(\theta\,\mathbf{r}^\wedge)(\mathbf{CM})\end{aligned} \tag{11}$$

T + **r**∧(**r**∧ **T**) is the orthogonal projection of **T** onto **r**. The formula of the double vector product leads to **T** + **r**∧(**r**∧ **T**) = (**T**· **r**) **r**. Then, we have:

$$\mathrm{M}' = \mathrm{C} + (\mathbf{T}\cdot\mathbf{r})\,\mathbf{r} + \exp(\theta\,\mathbf{r}^\wedge)(\mathbf{CM}) \tag{12}$$

We have established that any screw motion can be expressed with a system of geometric data: its axis (C, **r**), its angle θ and its vector translation (**T**· **r**) **r** along the axis. When the displacement is not a translation, one can also introduce the pitch p of the corresponding screw motion, p = (**T**· **r**) 2π /θ, or the reduced pitch k = p /2π. A frame-free characterization of a screw motion is:

$$\mathrm{M}' = \mathrm{C} + k\,\theta\,\mathbf{r} + \exp(\theta\,\mathbf{r}^\wedge)(\mathbf{CM}) \tag{13}$$

The characteristic geometric data of a given displacement, which is not a translation are: its axis (C, **r**), its angle θ; and its reduced pitch k. The vectors also change and any vector **V** becomes **V'** = exp(θ **r**^) **V**.

3. CONJUGATE OF A DISPLACEMENT

We begin with the special case of the conjugation in the group of spherical rotations around a given point. We deal also with the transformations of vectors in a displacement. The conjugate of exp(θ **r**^) by exp(ϕ **s**^) is exp(ϕ **s**^)exp(θ **r**^)[exp(ϕ **s**^)]$^{-1}$, which is equal to exp(ϕ **s**^) exp(θ **r**^) exp(- ϕ **s**^). Expanding exp(θ **r**^) in series, one proves that exp(ϕ**s**^)exp(θ**r**^)exp(-ϕ**s**^) is equal to exp{θ [exp(ϕ **s**^)(**r**^)exp(-ϕ **s**^)]}. Through further calculations, one can show that exp(ϕ **s**^) (**r**^) exp(-ϕ **s**^) is equal to **r'**^ with **r'** = exp(ϕ **s**^) **r**. Hence, the conjugate of exp(θ **r**^) is exp(θ **r'**^).

Let us consider now the conjugate of a spherical rotation about a given center O by a translation determined by the vector (**ON**)= N – O. It is the product of a translation of vector -(**ON**), which is the inverse of the translation of vector (**ON**), and the spherical rotation about O and the translation of vector (**ON**). Any point M becomes successively M_1, M_2 and M':

$$\begin{aligned}
&M \to M_1 = M - (\mathbf{ON}) \\
&M_1 \to M_2 = O + \exp(\theta\, \mathbf{r}^{\wedge})(\mathbf{OM}_1) \\
&M_2 \to M' = M_2 + (\mathbf{ON}) \\
&M \to M' = O + \exp(\theta\, \mathbf{r}^{\wedge})[(\mathbf{OM}) - (\mathbf{ON})] + (\mathbf{ON}) \\
&\quad = O + (\mathbf{ON}) - \exp(\theta\, \mathbf{r}^{\wedge})(\mathbf{ON}) + \exp(\theta\, \mathbf{r}^{\wedge})(\mathbf{OM})
\end{aligned} \tag{14}$$

This last expression depends on the origin O. It is important to notice that it can be written:

$$M \to M' = N + \exp(\theta\, \mathbf{r}^{\wedge})(\mathbf{NM}),$$

which is a spherical rotation about N with the Rodrigues vector θ**r**.

Two successive conjugations by two displacements result in a conjugation by the product of the two displacements. Therefore, the conjugate of a spherical rotation with the Rodrigues vector θ**r** about the point O by a displacement, which is the product of a rotation (ϕ **s**) followed by a translation (**ON**) will be the spherical rotation of Rodrigues vector θ **r'** about the point N, where **r'** = exp(ϕ **s**^) **r**.

Clearly, the conjugate of a translation with the vector **T** by another translation is the same translation of vector **T**. It is straightforward to establish that the conjugate of a translation of vector **T** by a spherical rotation about any given point with the Rodrigues vector ϕ**s**, is the translation of vector exp(ϕ **s**^) **T**.

Combining the foregoing two results, one can state that the conjugate of a translation **T** by a displacement that is the product of the rotation (ϕ **s**) about O and the translation (**ON**) is the translation of vector exp(ϕ **s**^) **T**. The conjugate of a product is the product of the conjugates. Consequently, the conjugate of the displacement

$$M \to M' = O + \mathbf{T} + \exp(\theta\, \mathbf{r}^{\wedge})(\mathbf{OM}) \tag{15}$$

by the displacement

$$M \to M' = O + (\mathbf{ON}) + \exp(\phi\, \mathbf{s}^{\wedge})(\mathbf{OM}) \tag{16}$$

is the displacement

$$\begin{aligned}
&M \to M' = N + \mathbf{T}' + \exp(\theta\, \mathbf{r}'^{\wedge})(\mathbf{NM}) \\
&\quad \begin{cases} N = O + (\mathbf{ON}) \\ \mathbf{r}' = \exp(\phi\, \mathbf{s}^{\wedge})\mathbf{r} \\ \mathbf{T}' = \exp(\phi\, \mathbf{s}^{\wedge})\mathbf{T} \end{cases}
\end{aligned} \tag{17}$$

The geometric elements N, **r'**, **T',** which characterize the conjugate of a given displacement are the transformed of the geometric elements O, **r**, **T** by the *conjugating* displacement.

When the same displacement is expressed using a point C belonging to its axis as an origin, the conjugate of the displacement

$$\mathrm{M} \to \mathrm{M}' = \mathrm{C} + k\theta\mathbf{r} + \exp(\theta\, \mathbf{r}^{\wedge})(\mathbf{CM}) \tag{18}$$

by the displacement

$$\mathrm{M} \to \mathrm{M}' = \mathrm{C} + (\mathbf{CC}') + \exp(\phi\, \mathbf{s}^{\wedge})(\mathbf{CM}) \tag{19}$$

is the displacement

$$\mathrm{M} \to \mathrm{M}' = \mathrm{C}' + k\theta\mathbf{r}' + \exp(\theta\, \mathbf{r}'^{\wedge})(\mathbf{C}'\mathbf{M}) \tag{20}$$

The angle θ and the pitch $2\pi k$ of a displacement are not changed by conjugation. The mutation concerns the axis when it exists: (C, **r**) becomes (C', **r'**); C' is the transformed of C by the conjugating displacement and **r'** is the transformed of **r** under the action of the vector transformation induced by the conjugating displacement. In a special way, two conjugate displacements can be regarded as congruent displacements.

The conjugation by a given displacement defines a mapping in the displacement set. Such a mapping is called an *inner automorphism*. It can be proven, in the case of the displacement group, that the set of inner automorphisms is isomorphic to the displacement group. Therefore, an inner automorphism can be likened to a displacement, which acts on displacements instead of a displacement acting on points.

4. THE SUBGROUPS OF THE DISPLACEMENT GROUP

A comprehensive list of the Lie subgroups of the Lie group of displacements was disclosed in [4]. However, the matrix representation of displacements involves cumbersome changes of frames of reference. Using the exponential of the skew-symmetric operator of the vector product, a coordinate-free notation of a displacement has been presented in [2]. Though it is claimed to be an intrinsic formulation, it is an origin-dependent approach. As aforesaid, a given displacement, which is not a translation, is determined by its axis characterized by (C, **r**), an angle θ and a reduced pitch k. In fact, the point C is a representative of a class of equivalent points. A point C' is equivalent to the point C if and only if $(\mathbf{CC'}) \wedge \mathbf{r} = \mathbf{0}$. A natural notation for any given displacement with an axis is $D = D(k, \theta, \mathrm{C}, \mathbf{r})$. From the datum of k, θ, C and **r**, a displacement can be readily expressed:

$$\mathrm{M} \to \mathrm{M}' = \mathrm{C} + k\theta\mathbf{r} + \exp(\theta\, \mathbf{r}^{\wedge})(\mathbf{C}'\mathbf{M}) = D(k,\theta,\mathrm{C},\mathbf{r})\mathrm{M} \tag{21}$$

When it is helpful, one can obtain the expression of this displacement employing any origin O by replacing C by O + (**OC**), (**CM**) by (**OM**) - (**OC**).

Based on the previous remarks, the subgroup of helical displacements of axis (N, **u**) and reduced pitch k can be denoted $\mathcal{H}(k)(\mathrm{N}, \mathbf{u})$. It is the set of the displacements $D(k, \alpha, \mathrm{N}, \mathbf{u})$ where the axis (N, **u**) and the reduced pitch k are given, while the angle α is the canonical parameter of this 1D set. The notation $\mathcal{H}(k)$ specifies the motion type, i. e., the conjugacy class of the helical motions with the pitch k. An element of the group $\mathcal{H}(k)(\mathrm{N}, \mathbf{u})$ can be designated by $H(k)(\mathrm{N}, \mathbf{u};$

α). $\mathcal{H}(k)(\text{N}, \mathbf{u})\}$ denotes the set $\{H(k)(\text{N},\mathbf{u};\ \alpha)|\alpha\in\mathbb{R})\}$, and $H(k)(\text{N}, \mathbf{u}; \alpha) \in \mathcal{H}(k)(\text{N}, \mathbf{u})$. Instead of k, the pitch $p = 2\pi k$ can be used. The subgroup may be denoted $\mathcal{H}(p)(\text{N}, \mathbf{u})$. Likewise, the group of rotations about a given axis (N, **u**) is denoted $\mathcal{R}(\text{N}, \mathbf{u})$. It is the special case with a zero pitch of the previous helical group: $\mathcal{R}(\text{N}, \mathbf{u}) = \mathcal{H}(0)(\text{N}, \mathbf{u})$.

The 1D group of rectilinear translations is $\mathcal{T}(\mathbf{u})$ where the unit vector **u** characterizes the direction of the possible translations.

The 2D group of cylindrical displacements, i.e., combinations of translations and rotations along a given axis (N, **u**) is $\mathcal{C}(\text{N}, \mathbf{u})$. It is the 2D set of all displacements having a given axis, with the two canonical parameters α and k.

The 2D group of planar translations is $\mathcal{T}(Pl)$, where Pl is a plane direction (or vector plane). Alternative designations of such a group are $\mathcal{T}(\perp\mathbf{u})$ and $\mathcal{T}_2(\mathbf{u})$ with $\mathbf{u} \perp Pl$. The 3D subgroup of spatial translations is $\mathcal{T}$.

A Y displacement has no name in the standard language. The group is made of the helical displacements with a given pitch $p = 2\pi k$ and axes parallel to **w** and the translations parallel to a plane orthogonal to **w**. A specified Y group is denoted $\mathcal{Y}(p)(\mathbf{w})$ or $\mathcal{Y}(k)(\mathbf{w})$. The group $\mathcal{G}(\mathbf{w})$ of planar gliding motions is the special case with a zero pitch: $\mathcal{G}(\mathbf{w})\} = \mathcal{Y}(0)(\mathbf{w})$. A Y motion was called a *pseudo-planar* motion.

The 3D group $\mathcal{S}(\text{N})$ of the rotations around all the axes passing through a given point N is called group of spherical rotations or spherical motions.

The mathematician Arthur Schönflies (also spelled Schoenflies) studied at some length a motion type [5], which combines spatial translation with rotation about axes with a given direction. He emphasized the analogy with the planar motion, without disclosing the group property of this 4-DoF motion. A set of these special motions is called a Schoenflies group and is denoted $\mathcal{X}(\mathbf{w})$, **w** being the unit vector parallel to the axes of the feasible rotations.

The previous groups are the proper subgroups of the group $\mathcal{D}$ of general displacements. $\mathcal{D}$ has two improper subgroups too. There are the group $\mathcal{D}$ itself and the zero-dimensional subgroup $\mathcal{E}$, which contains only the identity transform E. *To be a subgroup of a subgroup* is a binary relation of partial order as defined in abstract algebra. The displacement subgroups ordered by the foregoing relation were disclosed in [3-4].

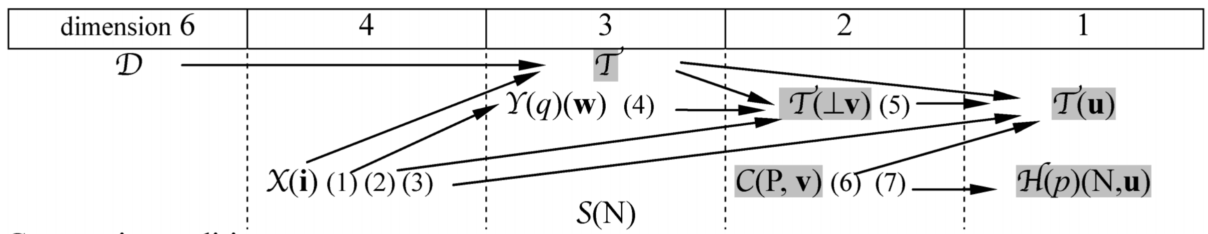
5. INVARIANT SUBGROUPS

If $\mathcal{G}_1$ is a subgroup of the group $\mathcal{D}$ of general displacements, and if $\mathcal{G}_2$ is a subgroup of $\mathcal{G}_1$, then one may consider conjugates of elements of $\mathcal{G}_2$ by elements of $\mathcal{G}_1$. If such conjugates belong also to $\mathcal{G}_2$, then $\mathcal{G}_2$ is globally invariant in the conjugation by the elements of $\mathcal{G}_1$. $\mathcal{G}_2$ is said to be an invariant (normal or distinguished) subgroup of $\mathcal{G}_1$. If $\mathcal{G}_3$ is a subgroup of $\mathcal{G}_2$, it is also a subgroup of $\mathcal{G}_1$; *to be a subgroup of* is a transitive binary relation; *to be an invariant subgroup of* is not transitive. It is worth noticing that any subgroup of $\mathcal{D}$ is invariant by self-conjugation; in other words, the relation *to be an invariant subgroup of* is reflexive.

As aforesaid, the conjugate of a displacement can be likened to a displaced displacement. Hence, it is easy to establish if a subgroup is invariant or not by detecting the invariance or the change of the geometric entities, which specify a subgroup in its conjugacy class. That way, the results shown in Table 1 can be readily obtained.

A group is said to be commutative or Abelian when the product does not depend on the factor order. It is quite evident that any subset of an Abelian group is invariant in the Abelian group. In Table 1, Abelian groups are designated in gray frames.

Table 1



Geometric conditions:
(1) $\mathbf{w} = \mathbf{i}, \forall q$; (2) $\mathbf{v} = \mathbf{i}$; (3) $\forall \mathbf{u} \perp \mathbf{i}$; (4) $\mathbf{v} = \mathbf{w}$; (5) $\forall \mathbf{u} \perp \mathbf{v}$; (6) $\mathbf{u} = \mathbf{v}$; (7) $\mathbf{u} = \mathbf{v}$, N $\in$ axis (P, $\mathbf{v}$)

If $\mathcal{S}$ is a subgroup of a group $\mathcal{G}$, and if g is a given element of $\mathcal{G}$, then the set $g\,\mathcal{S} = \{gs \mid s \in \mathcal{S}\}$ is called a left *coset* of $\mathcal{S}$ in the group $\mathcal{G}$. The elements of a left coset are equivalent. Because of the foregoing relation of equivalence, the left cosets of $\mathcal{S}$ provide a partition of $\mathcal{G}$. By the same token, right cosets can be defined. It can be proven that the left cosets (or the right cosets) of the subgroup $\mathcal{S}$ in the group $\mathcal{G}$ form a group *for the product of subsets* if and only if $\mathcal{S}$ is invariant in $\mathcal{G}$. Such a group of cosets is the *quotient* group $\mathcal{G}\,/\mathcal{S}$. This is a *set of subsets* of $\mathcal{G}$ and is not a subset of $\mathcal{G}$.

The problem is now to find an interpretation of the foregoing abstract concepts in the case of the displacement group $\mathcal{D}$. For example, the group $\mathcal{T}$ of spatial translations is an invariant subgroup of $\mathcal{D}$. What is the meaning of $\mathcal{D}\,/\mathcal{T}$? Let D_1 be any given displacement, a left coset of $\mathcal{T}$ in $\mathcal{D}$ is the set $D_1\mathcal{T}$. Because of the invariance of $\mathcal{T}$ in $\mathcal{D}$, we have $D_1\mathcal{T}(D_1)^{-1} = \mathcal{T}$, which implies $D_1\mathcal{T} = \mathcal{T}D_1$ for any D_1 belonging to $\mathcal{D}$. The coset $\mathcal{T}D_1$ is made of products of the displacement of D_1 with all the possible translations. A representative of the equivalence class of $\mathcal{T}D_1$ is a displacement having the translation vector $\mathbf{T} = \mathbf{0}$ at a given point N. We can define a mapping from the set of cosets into the set of displacements with the translation vector $\mathbf{0}$ at a given point N, which is the group $\mathcal{S}$(N) spherical rotations around N. This mapping is easily proven to be a group isomorphism. The quotient group $\mathcal{D}\,/\mathcal{T}$ is isomorphic to $\mathcal{S}$(N); $\mathcal{D}\,/\mathcal{T} \approx \mathcal{S}$(N), for any given point N. The quotient $\mathcal{D}\,/\mathcal{T}$ is the mathematical expression of the concept of orientation of a rigid body independently of its position. All the possible quotients between subgroups of $\mathcal{D}$ are isomorphic to displacement subgroups.

If a group $\mathcal{G}$ has two subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$ with $\mathcal{H}_1 \cap \mathcal{H}_2 = \{\text{identity}\}$, and if any element g of $\mathcal{G}$ is the commutative product of an element h_1 of $\mathcal{H}_1$ and an element h_2 of $\mathcal{H}_2$, namely $g = h_1h_2 = h_2h_1$, then it is straightforward to establish that $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant subgroups of $\mathcal{G}$ and $\mathcal{G}/\mathcal{H}_1 \approx \mathcal{H}_2$ and $\mathcal{G}/\mathcal{H}_2 \approx \mathcal{H}_1$. By hypothesis, $\mathcal{G}$ is equal to the products $\mathcal{H}_1\mathcal{H}_2 = \mathcal{H}_2\mathcal{H}_1$, but $\mathcal{G}$ is also isomorphic to the group called direct product of the groups $\mathcal{H}_1$ and $\mathcal{H}_2$, which is denoted $\mathcal{H}_1 \times \mathcal{H}_2$. The elements of $\mathcal{H}_1 \times \mathcal{H}_2$ are ordered pairs (h_1, h_2). The product in $\mathcal{H}_1 \times \mathcal{H}_2$ is defined by $(h_1, h_2) \times (h'_1, h'_2) = (h_1h'_1, h_2h'_2)$.

The quotients are presented in Table 2 together with the possible direct products.

6. KINEMATIC BONDS

A kinematic bond represents the coupling of two rigid bodies. The coupling results from the fact that both bodies belong to the same kinematic chain. In a kinematic chain that includes n rigid bodies, the number of kinematic bonds is $n\,(n - 1)/2$. A kinematic bond is the set of the feasible displacements of a body with respect to another body. Because a displacement is a

Table 2 Quotient groups

♦Quotient of $\mathcal{D}$ with respect to its invariant subgroup $\mathcal{T}$

$\mathcal{D} / \mathcal{T} \approx S(\mathrm{O}) \qquad \forall\, \mathrm{O}$

♦Quotient of $\mathcal{X}(\mathbf{i})$ with respect to its invariant subgroups

$\mathcal{X}(\mathbf{i}) / \mathcal{T} \approx \mathcal{H}(p)(\mathrm{N},\mathbf{i}) \qquad \forall\, \mathrm{N},\ \forall\, p$

$\mathcal{X}(\mathbf{i}) / \mathcal{Y}(q)(\mathbf{i}) \approx \mathcal{T}(\mathbf{i}) \approx \mathcal{H}(p)(\mathrm{N},\mathbf{i}) \quad \forall\, \mathrm{N}, \forall\, p \neq q$

$\mathcal{X}(\mathbf{i}) / \mathcal{T}(\perp \mathbf{i}) \approx \mathcal{C}(\mathrm{N},\mathbf{i}) \qquad \forall\, \mathrm{N}$

$\mathcal{X}(\mathbf{i}) / \mathcal{T}(\mathbf{i}) \approx \mathcal{Y}(q)(\mathbf{i}) \qquad \forall q$

• *possible direct products*

$\mathcal{X}(\mathbf{i}) = \mathcal{Y}(q)(\mathbf{i}) \times \mathcal{T}(\mathbf{i}) \quad \forall q$

♦Quotient of $\mathcal{Y}(q)(\mathbf{i})$ with respect to $\mathrm{T}(\perp \mathbf{i})$

$\mathcal{Y}(q)(\mathbf{i}) / \mathcal{T}(\perp \mathbf{i}) \approx \mathcal{H}(q)(\mathrm{N},\mathbf{i}) \quad \forall\, \mathrm{N}$

♦The $\mathcal{T}$ group

The $\mathcal{T}$ group is Abelian : any subset of $\mathcal{T}$ is invariant in $\mathcal{T}$.

$\mathcal{T} / \mathcal{T}(\perp \mathbf{k})\} \approx \mathcal{T}(\mathbf{u}) \quad \forall\ \mathbf{u} \neq \mathbf{k}$

$\mathcal{T} / \mathcal{T}(\mathbf{k}) \approx \mathcal{T}(\perp \mathbf{u}) \quad \forall\ \mathbf{u} \neq \mathbf{k}$

$\mathcal{T}(\perp \mathbf{k}) / \mathcal{T}(\mathbf{u})$ with $\mathbf{u} \perp \mathbf{k} \approx \mathcal{T}(\mathbf{v}) \quad \forall\, \mathbf{v} \neq \mathbf{u},\ \mathbf{v} \perp \mathbf{k}$

• *possible direct products*

$\mathcal{T} = \mathcal{T}(\mathbf{u}) \times \mathcal{T}(\mathbf{v})\} \times \mathcal{T}(\mathbf{w}) \qquad (\mathbf{u}, \mathbf{v}, \mathbf{w})$ is any vector base

$\mathcal{T}(\perp \mathbf{k}) = \mathcal{T}(\mathbf{u}) \times \mathcal{T}(\mathbf{v})\} \qquad \mathbf{u} \neq \mathbf{v},\ \forall\ \{\mathbf{u}, \mathbf{v}\} \perp \mathbf{k}$

♦The $\mathcal{C}(\mathrm{P},\mathbf{v})$ group

The $\mathcal{C}(\mathrm{P},\mathbf{v})$ group is Abelian : any subset of $\mathcal{C}(\mathrm{P},\mathbf{v})$ is invariant in $\mathcal{C}(\mathrm{P},\mathbf{v})$.

$\mathcal{C}(\mathrm{P},\mathbf{v}) / \mathcal{T}(\mathbf{v}) \approx \mathcal{H}(p)(\mathrm{N},\mathbf{v}) \quad \forall\, p, \forall\, \mathrm{N} \in \text{axis}\ (\mathrm{P}, \mathbf{v})$

$\mathcal{C}(\mathrm{P},\mathbf{v}) / \mathcal{H}(p)(\mathrm{N},\mathbf{v})$ with $\mathrm{N} \in \text{axis}\ (\mathrm{P}, \mathbf{v}) \approx \mathcal{T}(\mathbf{v}) \approx \mathcal{H}(p')(\mathrm{N}',\mathbf{v})\} \quad \forall\, p' \neq p,\ \forall\, \mathrm{N} \in \text{axis}\ (\mathrm{P}, \mathbf{v})$

• *possible direct products*

$\mathcal{C}(\mathrm{P},\mathbf{v}) = \mathcal{H}(p)(\mathrm{N},\mathbf{v}) \times \mathcal{H}(p')(\mathrm{N}',\mathbf{v}') \quad \forall\, p$ and p' with $p' \neq p, \forall\, \mathrm{N}$ and $\mathrm{N}' \in \text{axis}\ (\mathrm{P}, \mathbf{v})$

$\quad = \mathcal{T}(\mathbf{v}) \times \mathcal{H}(p)(\mathrm{N},\mathbf{v}) = \mathcal{H}(p)(\mathrm{N},\mathbf{v}) \times \mathcal{T}(\mathbf{v}) \quad \forall\ p, \forall\, \mathrm{N} \in \text{axis}\ (\mathrm{P}, \mathbf{v})$

change of rigid-body position, we have to begin with the description of the chain in a given initial posture.

An elementary case of chain is the serial arrangement of two kinematic pairs and the bond between the distal bodies is the composition product of the bonds generated by the pairs. A chain may be a single closed loop. Choosing two bodies in the loop, we obtain two serial chains, which connect in parallel the chosen bodies. The bond between these two bodies is the intersection of the bonds. It is clear that any kinematic chain can be analyzed by using the product and the intersection of kinematic bonds.

7. INVARIANCE OF KINEMATIC BONDS

The invariance of a kinematic bond will be explained with an example. $\mathcal{T}(\perp \mathbf{w})$ is an invariant subgroup of the group $\mathcal{Y}(q)(\mathbf{w})$. Let us consider the concatenation of a mechanical generator of the group $\mathcal{Y}(q)(\mathbf{w})$ and a mechanical generator of the $\mathcal{T}(\perp \mathbf{w})$ subgroup. The chain connects a fixed frame to an end body (Fig.1). The $\mathcal{Y}(q)(\mathbf{w})$ generator can be realized by three helical H pairs with axes parallel to $\mathbf{w}$ and the same pitch q. This HHH chain is a governing mechanism. That means that the kinematics pairs can be locked after an adjusting actuation. The generator of $\mathcal{T}(\perp \mathbf{w})$ is made of two prismatic P pairs, which are orthogonal to $\mathbf{w}$. When the governing mechanism is locked, the bond between the fixed base and the end body is $\mathcal{T}(\perp \mathbf{w})$ and this property is not changed after any adjusting motion in the governing HHH chain.

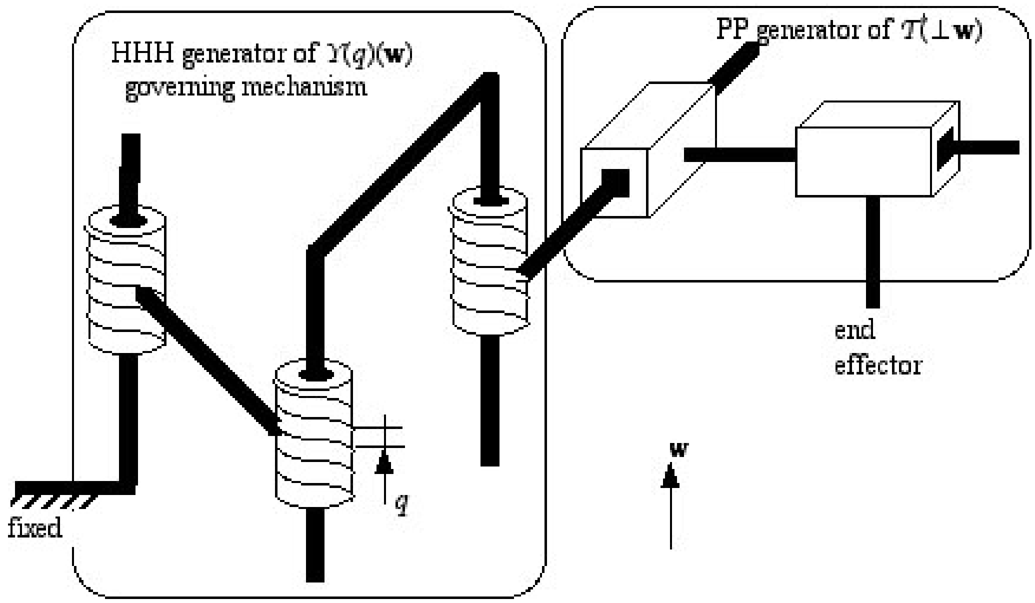


Fig. 1. Invariance of $\mathcal{T}(\perp \mathbf{w})$ in $\mathcal{Y}(q)(\mathbf{w})$.

8. REALIZATION OF A QUOTIENT SUBGROUP

The quotient $\mathcal{Y}(q)(\mathbf{w})/\ \mathcal{T}(\perp \mathbf{w})$ is isomorphic to the subgroup $\mathcal{H}(q)(\mathrm{N},\mathbf{w})$ of $\mathcal{Y}(q)(\mathbf{w})$, where N can be any point. This means that the concept of helical displacements $\mathcal{H}(q)(\mathrm{N},\mathbf{w})\}$ independent of the planar translations of $\mathcal{T}(\perp \mathbf{w})$ exists. In other respects, $\mathcal{Y}(q)(\mathbf{w})/\ \mathcal{T}(\perp \mathbf{w})$ can be considered as the set of equivalent subgroups $\mathcal{H}(q)(\mathrm{N'},\mathbf{w})\}$, where N' is a variable point belonging to the plane $\mathcal{T}(\perp \mathbf{w})$N. Such a concept is made concrete by a serial array of a PP generator of $\mathcal{T}(\perp \mathbf{w})$ followed by a helical H pair, which generates $\mathcal{H}(q)(\mathrm{N},\mathbf{w})$. This PPH chain generates helical displacements around any one of the axes (N',$\mathbf{w}$); N' $\in$plane $\mathcal{T}(\perp \mathbf{w})$N. The PP sub-chain governs the axis position. A HPP limb, which embodies $\mathcal{H}(q)(\mathrm{Q},\mathbf{w})\mathcal{T}(\perp \mathbf{w})\}$ and generates $\mathcal{Y}(q)(\mathbf{w})$ is added in parallel between the fixed frame and the moving end-effector, Fig. 2. When the governing PP subchain is locked, N' can not move. The end-effector motion is modeled by $\mathcal{H}(q)(\mathrm{Q},\mathbf{w})\mathcal{T}(\perp \mathbf{w})\cap\mathcal{H}(q)(\mathrm{N'},\mathbf{w}) = \mathcal{H}(q)(\mathrm{N'},\mathbf{w})$. The helical motion of the end effector can be actuated by the fixed H of $\mathcal{H}(q)(\mathrm{Q},\mathbf{w})$.

This chain is able to transmit a helical displacement from a fixed screw pair to another screw pair with the same pitch, which is movable by planar translation. The special case of two helical pairs of pitch $q = 0$ yields the Oldham shaft coupling [6].

9. KINEMATIC CHAINS DERIVED FROM DIRECT PRODUCTS

When a group can be decomposed into a direct product of two of its subgroups, one can synthesize single closed-loop chains with two types of movability that are uncoupled. For instance, let us consider the case of an $\mathcal{X}(\mathbf{i})$ group, which can be regarded as $\mathcal{Y}(q)(\mathbf{i}) \times \mathcal{T}(\mathbf{i})$. An 1-DoF single-loop chain associated with $\mathcal{Y}(q)(\mathbf{i})$ is made of four screw joints of pitch q with axes parallel to $\mathbf{i}$. An 1-DoF single-loop chain associated to $\mathcal{T}(\mathbf{i})$ is constituted by two prismatic pairs parallel to $\mathbf{i}$. These two loops can be imbricated as shown in Fig. 3 in a single closed loop, which is a 2-DoF kinematic chain associated to $\mathcal{X}(\mathbf{i})$. However, in this chain, the relative displacements in the screw pairs do not depend on the translations in the prismatic pairs, and vice versa.

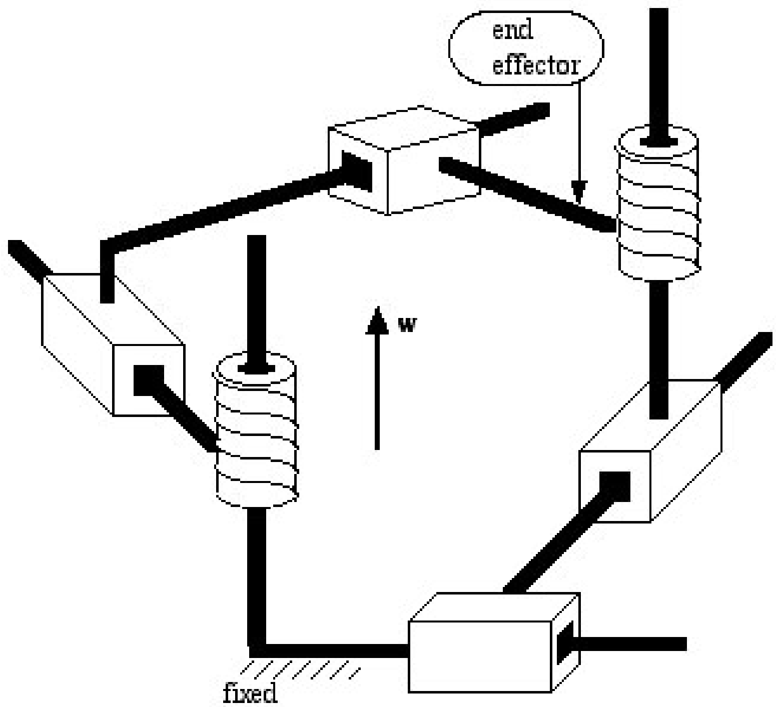


Fig. 2. Parallel arrangement of the embodiment of two elements of $\mathcal{Y}(q)(\mathbf{w})\ /\mathcal{T}(\perp\mathbf{w})$.

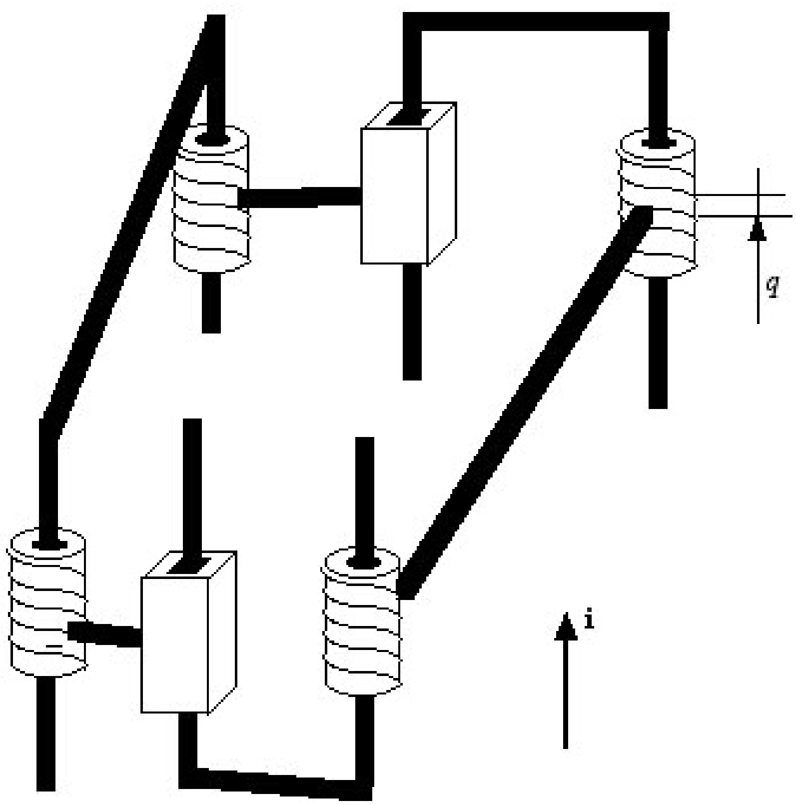


Fig. 3. Chain derived from $\mathcal{Y}(q)(\mathbf{i})\ \times\mathcal{T}(\mathbf{i})$.

10. CONCLUSION

Though the importance of the algebraic group structure of the set of displacements is recognized by an increasing number of authors in kinematics and mechanism science, the invariance by conjugation was not completely exploited. The concept of conjugation in the displacement group has been succinctly explained. Examples of kinematic chains with special mobility have been disclosed. As a last remark, the displacement group is a Lie group and its Lie algebra of screws is also capable of reflecting the invariance by conjugation, but the approach by Lie sub-algebras leads to the same results. Further applications of the group conjugation can be envisioned.

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