SKEW RAY TRACING AND ERROR ANALYSIS OF OPTICAL LENS WITH CYLINDRICAL BOUNDARY SURFACE

Te-Tan Liao[1], Shih-Hung Chen[2], Kuo-Ying Chen[3], Chun-Ta Chen[4]

[1] *Department of Mechanical Engineering, Far East University, Tainan, Taiwan, R. O. C.*
E-mail: ttliao@cc.feu.edu.tw

[2] *Department of Automation and Control Engineering, Far East University, Tainan, Taiwan, R. O. C.*

[3] *Department of Mechanical Engineering, Far East University, Tainan, Taiwan, R. O. C.*

[4] *Department of Mechanical and Automation Engineering, Da Yeh University, Changhua, Taiwan, R. O. C.*

Received February 2008, Accepted January 2009
No. 08-CSME-07, E.I.C. Accession 3045

ABSTRACT

This study applies computational geometric algebra based on a 4×4 homogeneous transformation matrix and Snell's law of geometrical optics to analyze skew rays and the errors of a light ray's path as it passes through a cylindrical lens. The author addresses two important topics: (1) the determination of the direction of a reflected or refracted ray by Snell's law and (2) the expression of the combination of two principal sources of light path error using error analysis. In topic (2), one of the sources is the translational errors Δd_{ix}, Δd_{iy}, and Δd_{iz} and the rotational errors $\Delta\omega_{ix}$, $\Delta\omega_{iy}$, and $\Delta\omega_{iz}$ that determine the deviation of the light path at each boundary surface, while the other source is the differential changes induced in the incident point position and the unit directional vector of the refracted/reflected ray as a result of differential changes in the position and unit directional vector of the light source. The methodology presented in this study provides a comprehensive and robust approach for evaluating the error of a light ray path as it passes through a cylindrical lens.

LANCER DE RAYONS OBLIQUES ET ANALYSE DE LENTILLES OPTIQUES À SURFACE CYLINDRIQUE FRONTIÈRE

RÉSUMÉ

Cette étude met en application l'algèbre géométrique de calcul basée sur une matrice de transformation homogène 4 X 4 et la loi de Snell sur l'optique géométrique pour analyser les rayons obliques, et les erreurs de trajectoire d'un lancer de rayons lumineux au moment de passer à travers une lentille cylindrique. L'auteur s'intéresse à deux points importants : (1) la détermination de la direction d'un rayon réfléchi ou réfracté selon la loi de Snell, et (2) l'expression de la combinaison de deux sources principales d'erreur de trajectoire de la lumière en utilisant la méthode d'analyse d'erreur. Au sujet du point (2), une des sources d'erreur ce sont les erreurs de translation Δd_{ix}, Δd_{iy}, et Δd_{iz} les erreurs de rotation $\Delta\omega_{ix}$, $\Delta\omega_{iy}$, et $\Delta\omega_{iz}$qui déterminent la déviation de la trajectoire de la lumière à chaque surface frontière. Tandis que l'autre source d'erreurs ce sont les changements différentiels provoqués dans la position du point d'incidence et le vecteur directionnel du rayon réfléchi ou réfracté, résultant des changements différentiels dans la position et le vecteur différentiel de la source lumineuse. La méthodologie présentée dans cette étude, propose une approche compréhensive solide pour l'évaluation de l'erreur de trajectoire d'un rayon lumineux au moment de passer à travers une lentille cylindrique.

1. INTRODUCTION

Currently, the ray tracing method is the core method used in many optical design tools and analysis software. The evaluation of the performance of an optical system during its theoretical design stage requires the ability to determine the paths of the light rays as they undergo reflection and refraction at the boundaries of the various optical elements within the system. The light path can be determined using a ray tracing technique in which the optical laws of reflection and refraction are systematically applied at each boundary encountered by the light ray [1,2]. In general, the light rays in an optical system can be classified as either axial rays, meridional rays, or skew rays. Axial rays and meridional rays can be traced using relatively simple trigonometric formulas or even graphically if a low precision is acceptable. Meridional rays in the paraxial region of an optical system can also be approximately traced via the successive application of matrix production [3]. A skew ray, which is the most general ray, is significantly difficult to trace. Nevertheless, skew ray tracing is essential while modeling an optical system and evaluating its performance. Consequently, when designing optical systems, it is conventional to trace the rays using actual light sources and the optical components being used. To facilitate the tracing of skew rays, Lin [4] reformulated the traditional optical laws of reflection and refraction in terms of revolution geometry. Subsequently, he conducted a sensitivity analysis based on a method of skew ray tracing to determine the changes in a light ray path as it crossed the boundary between the different media.

Ray tracing provides a powerful technique for analyzing the performance of optical lenses and is an essential task in the analysis and design of optical systems. The traced rays generally include those that start at a given set of object points and pass through a given set of points on the aperture stop [5]. In the differential ray tracing process, the effects of each optical component are evaluated by differentiating the equations related to the ray configuration of the rays before and after their transformation at a component surface [6,7]. Such ray tracing approaches enable the sensitivity of an optical system to design or manufacturing flaws to be assessed by correlating the differential changes of the reflected or refracted rays with the differential changes in the incident rays. Thus, the effect of each optical boundary surface on the overall ray path within the optical system can be systematically examined. Current commercial software for the analysis and design of optical systems use finite difference (FD) approximation methodology to estimate the gradient matrix of a ray with respect to the system variables. However, FD estimates are intrinsically inaccurate and are subject to a gross error when the denominator is excessively small with respect to the numerator. Lin and Tsai [8] avoid these problems and determine these gradients by the application of Snell's law. They provide information on the background and basics for determining the first-order gradients of skew rays of optical systems. Using these gradients, the differential vector of any ray can be estimated by determining the product of the developed gradient matrix and differential changes in the system variables. Furthermore, sensitivity analysis enables the construction of fundamental aberration functions, which greatly simplify the optical design task [9,10]. Lin and Liao [11] demonstrate the effectiveness of the sensitivity analysis approach by using a Jacobian matrix and the Newton-Raphson method to determine the path of the chief ray in a binocular stereovision system. Sensitivity analysis also enables the orientation of an image to be accurately determined. For example, by considering a prism design for illustration purposes, Tsai and Lin [12] applied the results of sensitivity analysis to construct a merit function describing the change in the orientation of an image as it was successively reflected/refracted at a series of flat boundary surfaces.

In many ways, the procedure used to design an optical system resembles that employed to design a mechanical machine tool. In the same way that optical systems comprise a structured series of optical elements, NC machine tools comprise a series of interconnected links and joints, each with its own unique mechanical properties and resolution characteristics. When designing a machine tool system, it is essential to identify the potential sources of error within the system and to clarify their individual and combined effects on the overall system performance so that the quality of the machined products can be reliably determined. In an earlier study [13], Tlusty showed that the overall error of an NC machine tool varies as a function of the combined effects of the individual errors in each of its six degrees of freedom. Ferreira and Liu [14] developed a model based on three rotational errors ($\Delta\Gamma_i$, $\Delta\Psi_i$, and $\Delta\Phi_i$) and three translational errors (ΔX_i, ΔY_i, and ΔZ_i) to enable the performance of a machine tool to be systematically examined. Considering the case of a jig-boring machine, Schultshik [15] applied these error definitions to investigate the combined effect of the various components of the volumetric accuracy of a machine tool on its machining precision.

The precision of the components produced by a machine tool system is dependent on the quality and configuration of each of its links and joints. Similarly, the image quality of an optical system is determined by the errors that inevitably exist in the fabrication and assembly of its individual optical components [16]. Therefore, by employing an approach similar to that used by Ferreira and Liu in analyzing the geometric errors of a machine tool [14], the current study develops a skew ray sensitivity analysis model based on six sources of light path errors, i.e., three rotational errors and three translational errors, to analyze the errors induced in a ray's light path as it is reflected and/or refracted by a succession of cylindrical boundary surfaces within an optical system. Significantly, in determining the deviation of the light path at each boundary surface, the proposed model incorporates an error matrix that takes account not only the effects of the rotational errors and translational errors but also the differential changes induced in the incident point position and unit directional vector of the refracted/reflected ray as a result of differential changes in the position and the unit directional vector of the light source. The present study achieves a more efficient cylindrical boundary skew ray tracing by the use of Snell's laws formulated as homogeneous transformation matrices, as described in Section 2. Error analysis is presented in Section 3. Numerical simulation of the errors of a refractive ray's light path as it passes through an optical element with a cylindrical boundary surface is presented in Section 4. The conclusion is presented in Section 5.

In the derivations performed in this paper, the position vector $P_{ix}i + P_{iy}j + P_{iz}k$ is written in the form of the column matrix ${}^{g}P_i = [P_{ix} \quad P_{iy} \quad P_{iz} \quad 1]^T$, where the pre-superscript "g" of the leading symbol ${}^{g}P_i$ indicates that the vector is referred with respect to the coordinate frame $(xyz)_g$. Furthermore, given a point ${}^{g}P_i$, its transformation ${}^{k}P_i$ is represented by the matrix product ${}^{k}P_i = {}^{k}A_g {}^{g}P_i$, where ${}^{k}A_g$ is a 4×4 matrix defining the position and orientation (referred to hereafter as the configuration) of the frame $(xyz)_g$ with respect to another frame $(xyz)_k$ [17]. The same notation rules are also applied to the unit directional vector ${}^{g}V_i = [V_{ix} \quad V_{iy} \quad V_{iz} \quad 0]^T$. Note that for the vectors referred with respect to the world frame $(xyz)_0$, the pre-superscript "0" is omitted for convenience.

2. SKEW RAY TRACING FOR CYLINDRICAL LENS

A fundamental feature of typical optical components is that they posses boundary surfaces that can be described by revolution geometry. Therefore, the first step in the determination of a typical optical system's skew ray path is the establishment of suitable revolution boundary surfaces.

A boundary surface ${}^{i}r_i$ at which reflection and refraction processes occur, can be obtained by the rotation of its generating curve $[x(\beta_i) \quad 0 \quad z(\beta_i) \quad 1]^T$ in the $x_i z_i$ plane about its optical z_i axis. i.e.,

$$ {}^{i}r_i = \begin{bmatrix} C\alpha_i & -S\alpha_i & 0 & 0 \\ S\alpha_i & C\alpha_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(\beta_i) \\ 0 \\ z(\beta_i) \\ 1 \end{bmatrix} = \begin{bmatrix} x(\beta_i)C\alpha_i \\ x(\beta_i)S\alpha_i \\ z(\beta_i) \\ 1 \end{bmatrix} \quad (1) $$

where C and S denote the cosine and sine functions, respectively. As shown in Fig. 1, ${}^{i}r_i$ for an optical element with cylindrical boundary surface can be obtained by rotating the generating curve ${}^{i}q_i = [R_i \quad 0 \quad h_i \quad 1]^T$ in the $x_i z_i$ plane about the z_i axis, i.e.,

$$ {}^{i}r_i = Rot(z_i,\alpha_i)\,{}^{i}q_i = \begin{bmatrix} C\alpha_i & -S\alpha_i & 0 & 0 \\ S\alpha_i & C\alpha_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_i \\ 0 \\ h_i \\ 1 \end{bmatrix} = [R_i C\alpha_i \quad R_i S\alpha_i \quad h_i \quad 1]^T, \quad (2) $$

where $Rot(z_i,\alpha_i)$ is the rotation transformation matrix about the z_i axis. Equation (2) provides a generic expression for parametrizing the boundaries of optical elements with cylindrical surfaces in terms of R_i and the polar angular position α_i. The unit normal vector ${}^{i}n_i$ along ${}^{i}r_i$ is given as

$$ {}^{i}n_i = s_i[C\alpha_i \quad S\alpha_i \quad 0 \quad 0]^T, \quad (3) $$

where s_i is set to +1 or –1 so that the cosine of the incident angle has a positive value, i.e., $C\theta_i > 0$.

Equations (2) and (3) give the parametric expression of ${}^{i}r_i$ and ${}^{i}n_i$ relative to frame $(xyz)_i$ built with respect to the i-th cylindrical boundary surface. However, many derivations in this paper are built relative to the world frame $(xyz)_0$. Therefore, we need the relative configuration of world frame $(xyz)_0$ with respect to frame $(xyz)_i$, which is given as the following 4×4 homogeneous transformation matrix:

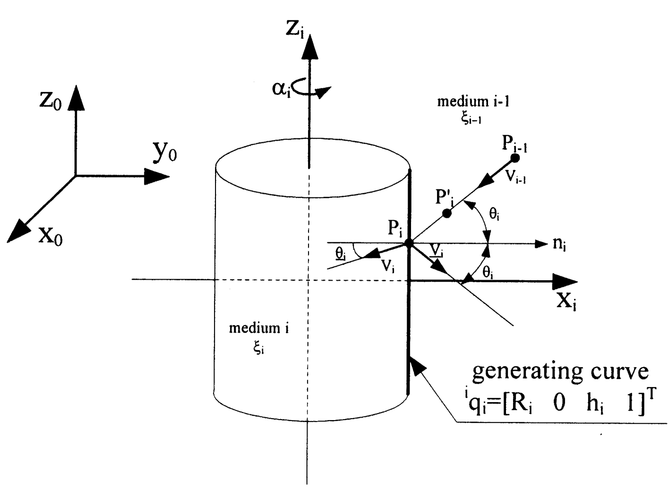

Fig. 1. Surface geometry and skew ray tracing for cylindrical optical element.

$$
{}^{i}A_0 = A_{i0} = \begin{bmatrix} I_{ix} & J_{ix} & K_{ix} & t_{ix} \\ I_{iy} & J_{iy} & K_{iy} & t_{iy} \\ I_{iz} & J_{iz} & K_{iz} & t_{iz} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4}
$$

where ${}^{i}A_0$ describes the position and orientation of the world frame $(xyz)_0$ with respect to the frame$(xyz)_i$. Vectors $[I_{ix} \quad I_{iy} \quad I_{iz} \quad 0]^T$, $[J_{ix} \quad J_{iy} \quad J_{iz} \quad 0]^T$, and $[K_{ix} \quad K_{iy} \quad K_{iz} \quad 0]^T$ describe the orientations of the three unit vectors of frame $(xyz)_0$ with respect to the frame $(xyz)_i$. The vector $[t_{ix} \quad t_{iy} \quad t_{iz} \quad 1]^T$ is the position vector of the origin of frame $(xyz)_0$ with respect to frame $(xyz)_i$ (see Fig. 2) [17].

The unit normal vector of the cylindrical boundary surface referred with respect to frame $(xyz)_0$, i.e. n_i, can be obtained by transforming ${}^{i}n_i$ to frame $(xyz)_0$, i.e.,

$$
n_i = \begin{bmatrix} n_{ix} \\ n_{iy} \\ n_{iz} \\ 0 \end{bmatrix} = {}^{0}A_i{}^{i}n_i = ({}^{i}A_0)^{-1}{}^{i}n_i = s_i \begin{bmatrix} I_{ix}C\alpha_i + I_{iy}S\alpha_i \\ J_{ix}C\alpha_i + J_{iy}S\alpha_i \\ K_{ix}C\alpha_i + K_{iy}S\alpha_i \\ 0 \end{bmatrix} \tag{5}
$$

In Fig. 1, a light ray originating at point $P_{i-1} = [P_{i-1x} \quad P_{i-1y} \quad P_{i-1z} \quad 1]^T$ and directed along the unit directional vector $V_{i-1} = [V_{i-1x} \quad V_{i-1y} \quad V_{i-1z} \quad 0]^T$ is reflected/refracted at the cylindrical medium boundary surface ${}^{i}r_i$. The parametric form of any intermediate point (denoted as P_i') along the incident ray is

$$
P_i' = [P_{i-1x} + V_{i-1x}\lambda \quad P_{i-1y} + V_{i-1y}\lambda \quad P_{i-1z} + V_{i-1z}\lambda \quad 1]^T, \tag{6}
$$

where $\lambda \geq 0$ is the magnitude of vector $P_{i\text{-}1}P_i'$. The parameter $\lambda = \lambda_i$ at which P_i' hits ${}^{i}r_i$ at the incident point

$$
P_i = [P_{i-1x} + V_{i-1x}\lambda_i \quad P_{i-1y} + V_{i-1y}\lambda_i \quad P_{i-1z} + V_{i-1z}\lambda_i \quad 1]^T \tag{7}
$$

is given by equating ${}^{i}r_i$ to ${}^{i}P_i'$ (${}^{i}P_i' = {}^{i}A_0 P_i'$ is the transformation of P_i' to frame $(xyz)_i$):

$$
\lambda_i = \frac{-D_i \pm \sqrt{D_i^2 - H_i \cdot E_i}}{H_i}. \tag{8}
$$

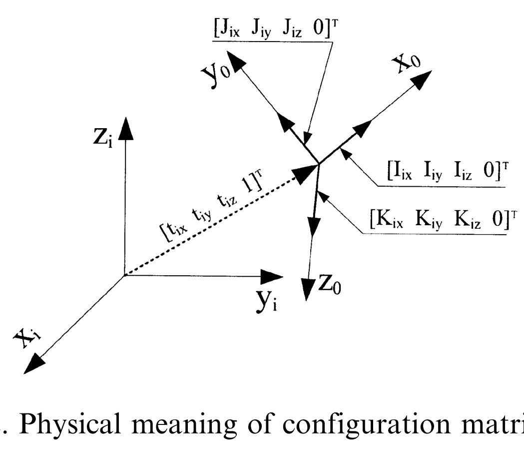

Fig. 2. Physical meaning of configuration matrix ${}^{i}A_0$.

Here, H_i, D_i and E_i are given by

$$H_i = G_{ix}^2 + G_{iy}^2, \tag{9}$$

$$D_i = G_{ix}(F_{ix} + t_{ix}) + G_{iy}(F_{iy} + t_{iy}), \tag{10}$$

$$E_i = (F_{ix} + t_{ix})^2 + (F_{iy} + t_{iy})^2 - R_i^2. \tag{11}$$

Where $F_{ix} = I_{ix}P_{i-1x} + J_{ix}P_{i-1y} + K_{ix}P_{i-1z}$, $F_{iy} = I_{iy}P_{i-1x} + J_{iy}P_{i-1y} + K_{iy}P_{i-1z}$, $F_{iz} = I_{iz}P_{i-1x} + J_{iz}P_{i-1y} + K_{iz}P_{i-1z}$, $G_{ix} = I_{ix}V_{i-1x} + J_{ix}V_{i-1y} + K_{ix}V_{i-1z}$, $G_{iy} = I_{iy}V_{i-1x} + J_{iy}V_{i-1y} + K_{iy}V_{i-1z}$, $G_{iz} = I_{iz}V_{i-1x} + J_{iz}V_{i-1y} + K_{iz}V_{i-1z}$.

The ambiguous sign of the root term in Eq. (8) indicates two possible points of intersection of the ray with a cylindrical boundary surface. Only one of these points is required in the current ray tracing analysis, i.e., the initial incident point, and the appropriate sign must therefore be chosen. The parameters α_i and h_i of the incident point on the cylindrical boundary surface are crucial in determining the subsequent path of the refracted ray and are defined as follows:

$$\alpha_i = \arctan(\rho_i, \sigma_i), \tag{12}$$

$$h_i = F_{iz} + \lambda_i G_{iz} + t_{iz}, \tag{13}$$

where $\sigma_i = F_{ix} + \lambda_i G_{ix} + t_{ix}$, $\rho_i = F_{iy} + \lambda_i G_{iy} + t_{iy}$.

To trace the reflected/refracted ray at the cylindrical boundary surface, we still need the incidence angle θ_i given by

$$C\theta_i = -V_{i-1}^T \cdot n_i = s_i(G_{ix}C\alpha_i + G_{iy}S\alpha_i). \tag{14}$$

The refraction angle $\underline{\theta}_i$ between these two optical media must satisfy Snell's law:

$$S\underline{\theta}_i = \frac{\xi_{i-1}}{\xi_i} S\theta_i = N_i S\theta_i, \tag{15}$$

where ξ_{i-1} and ξ_i are the indices of media i – 1 and i, respectively. $N_i = \xi_{i-1}/\xi_i$ is the relative refractive index of medium i – 1 with respect to medium i. The common unit normal vector m_i of n_i and V_{i-1} is given by

$$m_i = [\,m_{ix} \quad m_{iy} \quad m_{iz} \quad 0\,]^T = {n_i \times V_{i-1}}/{|n_i \times V_{i-1}|} = {n_i \times V_{i-1}}/{S\theta_i}. \tag{16}$$

According to the reflection (refraction) law of optics, the refracted (reflected) unit directional vector $V_i(\underline{V}_i)$ can be obtained by rotating n_i about m_i at an angle $\theta_p = \pi - \underline{\theta}_i$ ($\theta_p = \theta_i$). This leads to

$$V_i(\underline{V}_i) = [\,V_{ix} \quad V_{iy} \quad V_{iz} \quad 0\,]^T = \begin{bmatrix} m_{ix}^2(1-C\theta_P)+C\theta_P & m_{iy}m_{ix}(1-C\theta_P)-m_{iz}S\theta_P & m_{ix}m_{iz}(1-C\theta_P)+m_{iy}S\theta_p & 0 \\ m_{iy}m_{ix}(1-C\theta_P)+m_{iz}S\theta_P & m_{iy}^2(1-C\theta_P)+C\theta_P & m_{iz}m_{iy}(1-C\theta_P)-m_{ix}S\theta_P & 0 \\ m_{ix}m_{iz}(1-C\theta_P)-m_{iy}S\theta_P & m_{iz}m_{iy}(1-C\theta_P)+m_{ix}S\theta_p & m_{iz}^2(1-C\theta_P)+C\theta_P & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_{ix} \\ n_{iy} \\ n_{iz} \\ 0 \end{bmatrix}. \tag{17}$$

Further simplification of Eq. (17) is possible by utilizing the equation $S\theta_i m_i \times n_i = (n_i \times V_{i-1}) \times n_i = V_{i-1} - (n_i^T \cdot V_{i-1}) n_i = V_{i-1} + n_i C\theta_i$ (see Eq. (16)) and Snell's law $S\underline{\theta}_i = N_i S\theta_i$, resulting in:

$$V_i(\underline{V}_i) = \begin{bmatrix} V_{ix} \\ V_{iy} \\ V_{iz} \\ 0 \end{bmatrix} = \begin{bmatrix} n_{ix} C\theta_P + (n_{iz} m_{iy} - n_{iy} m_{iz}) S\theta_P \\ n_{iy} C\theta_P + (n_{ix} m_{iz} - n_{iz} m_{ix}) S\theta_P \\ n_{iz} C\theta_P + (n_{iy} m_{ix} - n_{ix} m_{iy}) S\theta_P \\ 0 \end{bmatrix} = \begin{bmatrix} n_{ix} C\theta_P + N_i (V_{i-1x} + n_{ix} C\theta_i) \\ n_{iy} C\theta_P + N_i (V_{i-1y} + n_{iy} C\theta_i) \\ n_{iz} C\theta_P + N_i (V_{i-1z} + n_{iz} C\theta_i) \\ 0 \end{bmatrix}. \quad (18)$$

From Eq. (18), the refracted unit direction V_i ($\theta_p = \pi - \underline{\theta}_i$) is expressed as a unit vector as

$$V_i = \begin{bmatrix} V_{ix} \\ V_{iy} \\ V_{iz} \\ 0 \end{bmatrix} = \begin{bmatrix} -n_{ix} \sqrt{1 - N_i^2 + (N_i C\theta_i)^2} + N_i (V_{i-1x} + n_{ix} C\theta_i) \\ -n_{iy} \sqrt{1 - N_i^2 + (N_i C\theta_i)^2} + N_i (V_{i-1y} + n_{iy} C\theta_i) \\ -n_{iz} \sqrt{1 - N_i^2 + (N_i C\theta_i)^2} + N_i (V_{i-1z} + n_{iz} C\theta_i) \\ 0 \end{bmatrix}, \quad (19)$$

while the reflected unit direction $\underline{V}_i$ ($\theta_p = \theta_i$ and $N_i = 1$) is given by

$$\underline{V}_i = \begin{bmatrix} \underline{V}_{ix} \\ \underline{V}_{iy} \\ \underline{V}_{iz} \\ 0 \end{bmatrix} = \begin{bmatrix} V_{i-1x} + 2 n_{ix} C\theta_i \\ V_{i-1y} + 2 n_{iy} C\theta_i \\ V_{i-1z} + 2 n_{iz} C\theta_i \\ 0 \end{bmatrix}. \quad (20)$$

Following refraction (reflection), the light ray proceeds with point P_i as its new point of origin and V_i as its new unit directional vector. We have illustrated how the path of the reflected or refracted ray relative to a single cylindrical boundary surface can be found. We can apply this approach successively to trace rays relative to a series of optical elements that possess only cylindrical boundary surfaces.

3. ERROR ANALYSIS FOR CYLINDRICAL LENS

In optical systems, errors inevitably occur between the designed position and orientation of the optical elements and their actual position and orientation. In analyzing these errors, the relative positions and orientations of $(xyz)_0$ with respect to the ideal frame $(xyz)_i$ and the actual frame $(xyz)_a$ can be expressed, respectively, as

$$^{i}A_0 = \begin{bmatrix} I_{ix} & J_{ix} & K_{ix} & t_{ix} \\ I_{iy} & J_{iy} & K_{iy} & t_{iy} \\ I_{iz} & J_{iz} & K_{iz} & t_{iz} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (21)$$

$$^{a}A_0 = \begin{bmatrix} I_{ax} & J_{ax} & K_{ax} & t_{ax} \\ I_{ay} & J_{ay} & K_{ay} & t_{ay} \\ I_{az} & J_{az} & K_{az} & t_{az} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (22)$$

As shown in Fig. 3, the position and orientation errors of any element with a cylindrical boundary surface within the optical system can be described in terms of three translational errors of the origin of frame $(xyz)_a$, i.e., Δd_{ix}, Δd_{iy}, and Δd_{iz}, and three rotational errors of the

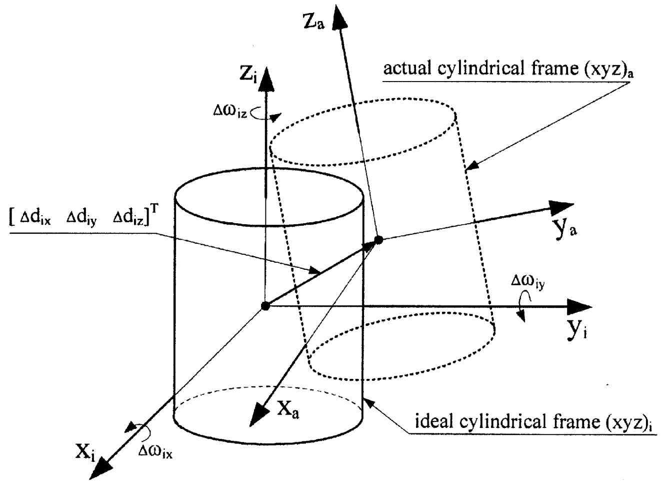

Fig. 3. Definition of translational and rotational errors of coordinated frame of the cylindrical boundary surface.

three axes of frame $(xyz)_a$ with respect to frame $(xyz)_i$, i.e., $\Delta\omega_{ix}$, $\Delta\omega_{iy}$, and $\Delta\omega_{iz}$. The overall effect of these six errors can be mathematically expressed using a matrix ${}^iA_a$ of the form

$$\begin{aligned}{}^iA_a &= \mathrm{Trans}(\Delta d_{ix},\Delta d_{iy},\Delta d_{iz})\mathrm{Rot}(z,\Delta\omega_{iz})\mathrm{Rot}(y,\Delta\omega_{iy})\mathrm{Rot}(x,\Delta\omega_{iz})\\ &= \begin{bmatrix} C\Delta\omega_{iz}C\Delta\omega_{iy} & C\Delta\omega_{iz}S\Delta\omega_{iy}S\Delta\omega_{ix}-S\Delta\omega_{iz}C\Delta\omega_{ix} & C\Delta\omega_{iz}S\Delta\omega_{iy}C\Delta\omega_{ix}+S\Delta\omega_{iz}S\Delta\omega_{ix} & \Delta d_{ix}\\ S\Delta\omega_{iz}C\Delta\omega_{iy} & S\Delta\omega_{iz}S\Delta\omega_{iy}S\Delta\omega_{ix}+C\Delta\omega_{iz}C\Delta\omega_{ix} & S\Delta\omega_{iz}S\Delta\omega_{iy}C\Delta\omega_{ix}-C\Delta\omega_{iz}S\Delta\omega_{ix} & \Delta d_{iy}\\ -S\Delta\omega_{iy} & C\Delta\omega_{iy}S\Delta\omega_{ix} & C\Delta\omega_{iy}C\Delta\omega_{ix} & \Delta d_{iz}\\ 0 & 0 & 0 & 1\end{bmatrix}.\end{aligned} \tag{23}$$

Since in an optical system, the translational and rotational errors are small, Eq. (23) can be approximated by the first-order Taylor series expansion and rewritten in the form

$${}^iA_a = \begin{bmatrix} \bar{I}_{ix} & \bar{J}_{ix} & \bar{K}_{ix} & \bar{t}_{ix}\\ \bar{I}_{iy} & \bar{J}_{iy} & \bar{K}_{iy} & \bar{t}_{iy}\\ \bar{I}_{iz} & \bar{J}_{iz} & \bar{K}_{iz} & \bar{t}_{iz}\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix} 1 & -\Delta\omega_{iz} & \Delta\omega_{iy} & \Delta d_{ix}\\ \Delta\omega_{iz} & 1 & -\Delta\omega_{ix} & \Delta d_{iy}\\ -\Delta\omega_{iy} & \Delta\omega_{ix} & 1 & \Delta d_{iz}\\ 0 & 0 & 0 & 1\end{bmatrix}. \tag{24}$$

Applying the assumption ${}^aA_0 = {}^iA_0 + d\,{}^iA_0 = {}^iA_a^{-1}\,{}^iA_0$, it can be shown that

$$\begin{aligned}{}^aA_0 &= \begin{bmatrix} I_{ix}+\Delta I_{ix} & J_{ix}+\Delta J_{ix} & K_{ix}+\Delta K_{ix} & t_{ix}+\Delta t_{ix}\\ I_{iy}+\Delta I_{iy} & J_{iy}+\Delta J_{iy} & K_{iy}+\Delta K_{iy} & t_{iy}+\Delta t_{iy}\\ I_{iz}+\Delta I_{iz} & J_{iz}+\Delta J_{iz} & K_{iz}+\Delta K_{iz} & t_{iz}+\Delta t_{iz}\\ 0 & 0 & 0 & 1\end{bmatrix}\\ &= \begin{bmatrix} 1 & \Delta\omega_{iz} & -\Delta\omega_{iy} & -\Delta d_{ix}\\ -\Delta\omega_{iz} & 1 & \Delta\omega_{ix} & -\Delta d_{iy}\\ \Delta\omega_{iy} & -\Delta\omega_{ix} & 1 & -\Delta d_{iz}\\ 0 & 0 & 0 & 1\end{bmatrix} \begin{bmatrix} I_{ix} & J_{ix} & K_{ix} & t_{ix}\\ I_{iy} & J_{iy} & K_{iy} & t_{iy}\\ I_{iz} & J_{iz} & K_{iz} & t_{iz}\\ 0 & 0 & 0 & 1\end{bmatrix}.\end{aligned} \tag{25}$$

From Eq. (25), it can be deduced that

$$\begin{cases}\Delta I_{ix}=I_{iy}\Delta\omega_{iz}-I_{iz}\Delta\omega_{iy}\\ \Delta I_{iy}=-I_{ix}\Delta\omega_{iz}+I_{iz}\Delta\omega_{ix}\,,\\ \Delta I_{iz}=I_{ix}\Delta\omega_{iy}-I_{iy}\Delta\omega_{ix}\end{cases}\qquad \begin{cases}\Delta J_{ix}=J_{iy}\Delta\omega_{iz}-J_{iz}\Delta\omega_{iy}\\ \Delta J_{iy}=-J_{ix}\Delta\omega_{iz}+J_{iz}\Delta\omega_{ix}\,,\\ \Delta J_{iz}=J_{ix}\Delta\omega_{iy}-J_{iy}\Delta\omega_{ix}\end{cases}$$

$$\begin{cases}\Delta K_{ix}=K_{iy}\Delta\omega_{iz}-K_{iz}\Delta\omega_{iy}\\ \Delta K_{iy}=-K_{ix}\Delta\omega_{iz}+K_{iz}\Delta\omega_{ix}\,,\\ \Delta K_{iz}=K_{ix}\Delta\omega_{iy}-K_{iy}\Delta\omega_{ix}\end{cases}\qquad \begin{cases}\Delta t_{ix}=t_{iy}\Delta\omega_{iz}-t_{iz}\Delta\omega_{iy}-\Delta d_{ix}\\ \Delta t_{iy}=-t_{ix}\Delta\omega_{iz}+t_{iz}\Delta\omega_{ix}-\Delta d_{iy}\,.\\ \Delta t_{iz}=t_{ix}\Delta\omega_{iy}-t_{iy}\Delta\omega_{ix}-\Delta d_{iz}\end{cases}$$

Furthermore, differentiating Eqs. (7), (19), and (20), it can be shown that the differential changes in the incident point position ΔP_i, ΔV_i, and vector $\Delta\underline{V}_i$ are respectively given by

$$\Delta P_i=\Delta P_{i-1}+\lambda_i\Delta V_{i-1}+V_{i-1}\Delta\lambda_i=\underline{M}_{P_i}\begin{bmatrix}\Delta P_{i-1}\\ \Delta V_{i-1}\end{bmatrix}+\underline{\underline{M}}_{P_i}[\Delta e_i], \tag{26}$$

$$\Delta V_i=\underline{M}_{V_i}\begin{bmatrix}\Delta P_{i-1}\\ \Delta V_{i-1}\end{bmatrix}+\underline{\underline{M}}_{V_i}[\Delta e_i], \tag{27}$$

and

$$\Delta\underline{V}_i=\underline{M}_{\underline{V}_i}\begin{bmatrix}\Delta P_{i-1}\\ \Delta V_{i-1}\end{bmatrix}+\underline{\underline{M}}_{\underline{V}_i}[\Delta e_i]. \tag{28}$$

Combining Eqs. (26), (27), and (28), the differential changes in ΔP_i and the refracted (reflected) ray unit directional vectors $\Delta V_i(\Delta\underline{V}_i)$ can be derived as

$$\begin{bmatrix}\Delta P_i\\ \Delta V_i\end{bmatrix}=\underline{M}_i\begin{bmatrix}\Delta P_{i-1}\\ \Delta V_{i-1}\end{bmatrix}+\underline{\underline{M}}_i[e_i], \tag{29}$$

where $[e_i]=[\,\Delta d_{ix}\quad \Delta d_{iy}\quad \Delta d_{iz}\quad \Delta\omega_{ix}\quad \Delta\omega_{iy}\quad \Delta\omega_{iz}\,]^T$ and the forms of $\underline{M}_i$ and $\underline{\underline{M}}_i$ are presented in the APPENDIX. The corresponding light path error induced at the (n – 1)-th boundary surface can then be determined from

$$\begin{aligned}\begin{bmatrix}\Delta P_{n-1}\\ \Delta V_{n-1}\end{bmatrix}&=\underline{\underline{M}}_{n-1}[e_{n-1}]+\underline{M}_{n-1}\begin{bmatrix}\Delta P_{n-2}\\ \Delta V_{n-2}\end{bmatrix}\\ &=\underline{\underline{M}}_{n-1}[e_{n-1}]+\underline{M}_{n-1}\underline{\underline{M}}_{n-2}[e_{n-2}]+\underline{M}_{n-1}\underline{M}_{n-2}\begin{bmatrix}\Delta P_{n-3}\\ \Delta V_{n-3}\end{bmatrix}\\ &=\underline{\underline{M}}_{n-1}[e_{n-1}]+\underline{M}_{n-1}\underline{\underline{M}}_{n-2}[e_{n-2}]+\underline{M}_{n-1}\underline{M}_{n-2}\underline{\underline{M}}_{n-3}[e_{n-3}]+\cdots\cdots\cdots\\ &\quad+\underline{M}_{n-1}\underline{M}_{n-2}\underline{M}_{n-3}\underline{M}_{n-4}\cdots\cdots\underline{M}_2\underline{\underline{M}}_1[e_1]\\ &=M_{n-1}[e_{n-1}]+M_{n-2}[e_{n-2}]+M_{n-3}[e_{n-3}]+\cdots+M_2[e_2]+M_1[e_1].\end{aligned} \tag{30}$$

In Eq. (30), M_i $(i=1\sim n-1)$ is an error analysis matrix of the i-th boundary surface r_i that can be used to analyze the variation of the exit ray of the optical system. Moreover, M_i is combined the ray path errors at the i-th boundary surface (i.e. three translational errors and three rotational errors) with the differential changes induced in the reflected/refracted ray unit directional vector and incident point by differential changes in the light source and unit directional vector of the incident ray, i.e.

$$M_i = \begin{bmatrix} \frac{\partial P_{ix}}{\partial d_{ix}} & \frac{\partial P_{ix}}{\partial d_{iy}} & \frac{\partial P_{ix}}{\partial d_{iz}} & \frac{\partial P_{ix}}{\partial \omega_{ix}} & \frac{\partial P_{ix}}{\partial \omega_{iy}} & \frac{\partial P_{ix}}{\partial \omega_{iz}} \\ \frac{\partial P_{iy}}{\partial d_{ix}} & \frac{\partial P_{iy}}{\partial d_{iy}} & \frac{\partial P_{iy}}{\partial d_{iz}} & \frac{\partial P_{iy}}{\partial \omega_{ix}} & \frac{\partial P_{iy}}{\partial \omega_{iy}} & \frac{\partial P_{iy}}{\partial \omega_{iz}} \\ \frac{\partial P_{iz}}{\partial d_{ix}} & \frac{\partial P_{iz}}{\partial d_{iy}} & \frac{\partial P_{iz}}{\partial d_{iz}} & \frac{\partial P_{iz}}{\partial \omega_{ix}} & \frac{\partial P_{iz}}{\partial \omega_{iy}} & \frac{\partial P_{iz}}{\partial \omega_{iz}} \\ \frac{\partial V_{ix}}{\partial d_{ix}} & \frac{\partial V_{ix}}{\partial d_{iy}} & \frac{\partial V_{ix}}{\partial d_{iz}} & \frac{\partial V_{ix}}{\partial \omega_{ix}} & \frac{\partial V_{ix}}{\partial \omega_{iy}} & \frac{\partial V_{ix}}{\partial \omega_{iz}} \\ \frac{\partial V_{iy}}{\partial d_{ix}} & \frac{\partial V_{iy}}{\partial d_{iy}} & \frac{\partial V_{iy}}{\partial d_{iz}} & \frac{\partial V_{iy}}{\partial \omega_{ix}} & \frac{\partial V_{iy}}{\partial \omega_{iy}} & \frac{\partial V_{iy}}{\partial \omega_{iz}} \\ \frac{\partial V_{iz}}{\partial d_{ix}} & \frac{\partial V_{iz}}{\partial d_{iy}} & \frac{\partial V_{iz}}{\partial d_{iz}} & \frac{\partial V_{iz}}{\partial \omega_{ix}} & \frac{\partial V_{iz}}{\partial \omega_{iy}} & \frac{\partial V_{iz}}{\partial \omega_{iz}} \end{bmatrix}. \tag{31}$$

4. SIMULATION OF REFRACTIVE RAY PATH ERRORS AT CYLINDRICAL LENS

This section demonstrates the validity of the proposed error analysis methodology using the case of optical element with refractive cylindrical boundary surface for illustration purposes. Error analysis using the proposed methodology yields a direct and rapid analytical expression of the errors of a light ray's path as it passes through optical elements with cylindrical boundary surfaces. In the current analysis, it is assumed that the light source is located at $P_0 = [0 \quad P_{0y} \quad 0 \quad 1]^T$. The unit directional vector of the incoming ray $V_0 = [C\beta_0 C\alpha_0 \quad C\beta_0 S\alpha_0 \quad S\beta_0 \quad 0]^T$ (α_0 and β_0 are the polar angles of the incoming ray) is refracted by the cylindrical boundary surface. The polar angles of the incoming ray are $\alpha_0 = \beta_0 = 0$ and $\alpha_0 = 0^o$, $\beta_0 = 15^o$. The radius of the cylindrical boundary surface is $R_1 = 22.54$mm

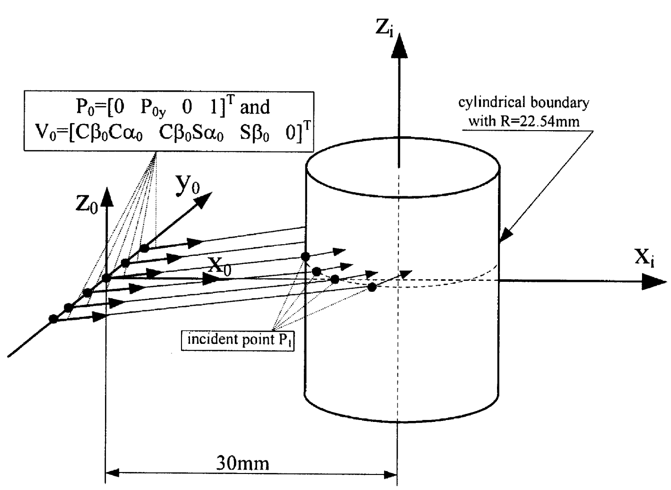

Fig. 4. Refraction at cylindrical boundary.

and the refractive index ratios are $N_1 = 1.5$, $N_1 = 2.0$, and $N_1 = 2.5$ (see Fig. 4). The relative configuration of $(xyz)_0$ with respect to cylindrical boundary surface frame $(xyz)_1$ is given by

$$^{1}A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -30 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{32}$$

In general, the translational error Δd_{iz} and the rotational error $\Delta\omega_{iz}$ do not influence the position of the exit ray, and hence the variations $\frac{\partial P_i}{\partial d_{iz}}(\frac{\partial V_i}{\partial d_{iz}})$ and $\frac{\partial P_i}{\partial \omega_{iz}}(\frac{\partial V_i}{\partial \omega_{iz}})$ are equal to zero. Then, the error analysis matrix M_i is given by

$$M_i = \begin{bmatrix} \frac{\partial P_{ix}}{\partial d_{ix}} & \frac{\partial P_{ix}}{\partial d_{iy}} & 0 & \frac{\partial P_{ix}}{\partial \omega_{ix}} & \frac{\partial P_{ix}}{\partial \omega_{iy}} & 0 \\ \frac{\partial P_{iy}}{\partial d_{ix}} & \frac{\partial P_{iy}}{\partial d_{iy}} & 0 & \frac{\partial P_{iy}}{\partial \omega_{ix}} & \frac{\partial P_{iy}}{\partial \omega_{iy}} & 0 \\ \frac{\partial P_{iz}}{\partial d_{ix}} & \frac{\partial P_{iz}}{\partial d_{iy}} & 0 & \frac{\partial P_{iz}}{\partial \omega_{ix}} & \frac{\partial P_{iz}}{\partial \omega_{iy}} & 0 \\ \frac{\partial V_{ix}}{\partial d_{ix}} & \frac{\partial V_{ix}}{\partial d_{iy}} & 0 & \frac{\partial V_{ix}}{\partial \omega_{ix}} & \frac{\partial V_{ix}}{\partial \omega_{iy}} & 0 \\ \frac{\partial V_{iy}}{\partial d_{ix}} & \frac{\partial V_{iy}}{\partial d_{iy}} & 0 & \frac{\partial V_{iy}}{\partial \omega_{ix}} & \frac{\partial V_{iy}}{\partial \omega_{iy}} & 0 \\ \frac{\partial V_{iz}}{\partial d_{ix}} & \frac{\partial V_{iz}}{\partial d_{ix}} & 0 & \frac{\partial V_{iz}}{\partial \omega_{ix}} & \frac{\partial V_{iz}}{\partial \omega_{iy}} & 0 \end{bmatrix}. \tag{33}$$

Figure 5 illustrates the variations in $\partial P_{1x}/\partial V_{0y}$ and $\partial P_{1y}/\partial V_{0y}$ with the light source position P_{0y} at the refractive index ratio $N_1 = 1.5$. In Fig. 5, it can be observed that the variation in $\partial P_{1x}/\partial V_{0y}$ is antisymmetric with regard to P_{0y}, while the variation in $\partial P_{1y}/\partial V_{0y}$ is symmetric with regard to P_{0y}. Moreover, total reflection occurs at a small value of P_{0y} for the incoming ray $V_0 = [\cos 15^o \quad 0 \quad \sin 15^o \quad 0]^T$. Figure 6 illustrates the variations in $\partial V_{1x}/\partial d_{1y}$ and $\partial V_{1y}/\partial d_{1y}$ with the light source position P_{0y} at the refractive index ratio $N_1 = 1.5$. From figure 6, it can be observed that the variation in $\partial V_{1x}/\partial d_{1y}$ is antisymmetric with regard to P_{0y}, while the variation in $\partial V_{1y}/\partial d_{1y}$ is symmetric with regard to P_{0y}.

Figure 7 illustrates the variations in $\partial V_{1y}/\partial \omega_{1x}$ and $\partial V_{1y}/\partial \omega_{1y}$ with P_{0y} as a function of the refractive index ratio N_1 for $V_0 = [\cos 15^o \quad 0 \quad \sin 15^o \quad 0]^T$. From Fig. 7, it can be observed that the variation in $\partial V_{1y}/\partial \omega_{1x}$ is symmetric with regard to P_{0y}, while the variation in $\partial V_{1y}/\partial \omega_{1y}$ is antisymmetric with respect to P_{0y}. In Fig. 7, it can be observed that as the value of N_1 increases, the total reflection occurs at smaller values of P_{0y}. Figure 8 illustrates the variation in $\partial V_{1z}/\partial \omega_{1x}$ with P_{0y} as a function of N_1 for $V_0 = [\cos 15^o \quad 0 \quad \sin 15^o \quad 0]^T$. From Fig. 8, it can be observed that the variation in $\partial V_{1z}/\partial \omega_{1x}$ is antisymmetric with respect to P_{0y}.

In general, Figs. 5~8 demonstrate that the sensitivity and error analysis of the boundary surface frame result in significant deviations in the orientation of the light ray refracted by the optical element with cylindrical surface. If the sensitivity and error analysis matrix of boundary surface frame $(xyz)_i$ are known, we can obtain the variation in the path of the exit light ray of an optical system that occurs due to differential changes in the six-degrees of freedom (three translational and three rotational) of the boundary surface frame $(xyz)_i$ by using the continuous

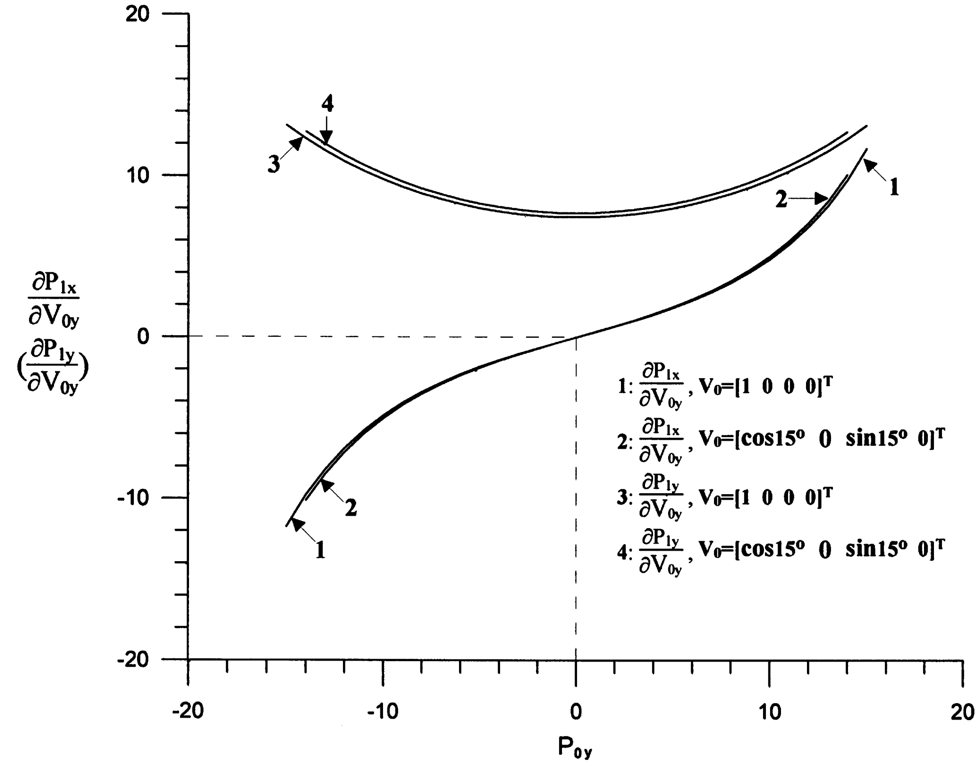

Fig. 5. Variations in $\partial P_{1x}/\partial V_{0y}$ and $\partial P_{1y}/\partial V_{0y}$ with P_{0y} for $N_1 = 1.5$.

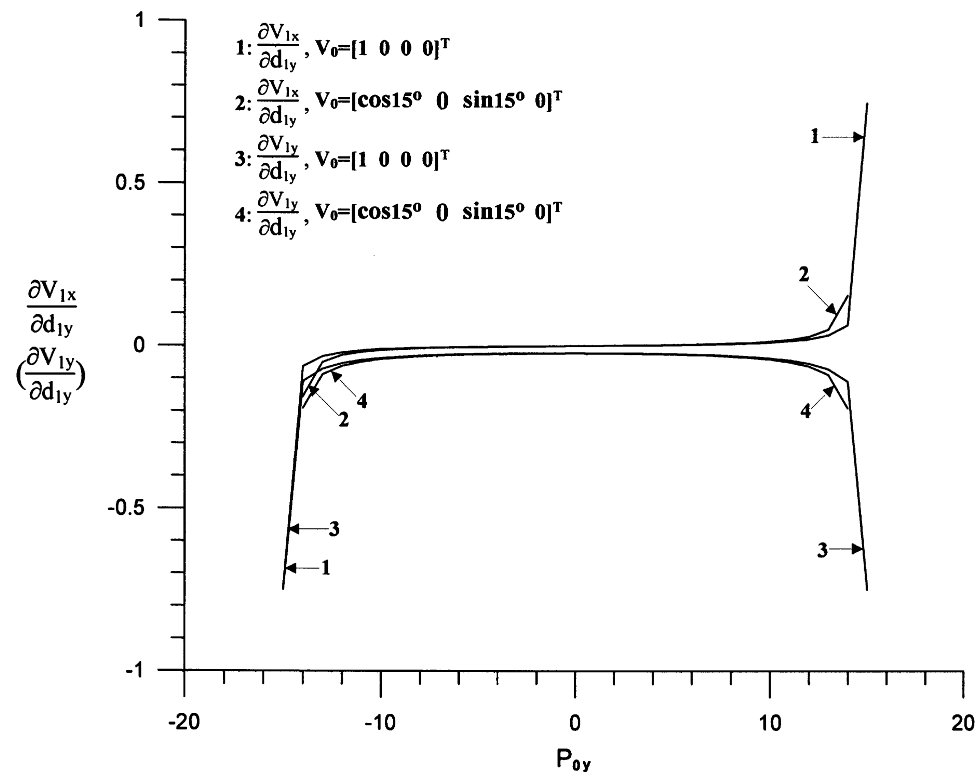

Fig. 6. Variations in $\partial V_{1x}/\partial d_{1y}$ and $\partial V_{1y}/\partial d_{1y}$ with P_{0y} for $N_1 = 1.5$.

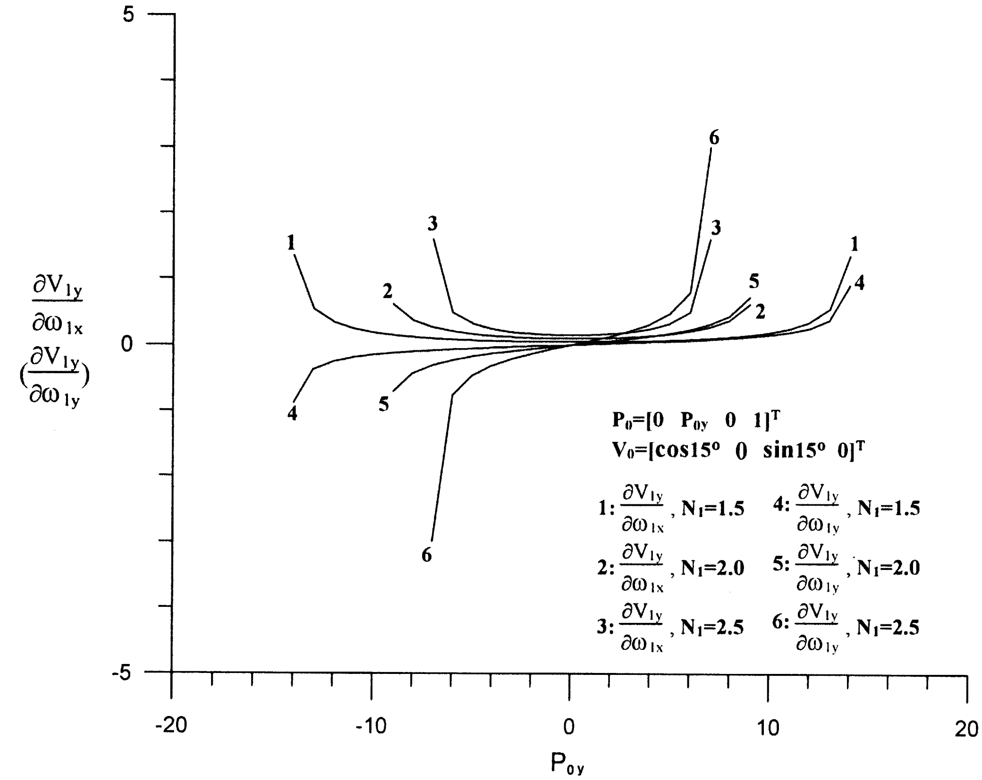

Fig. 7. Variations in $\partial V_{1y}/\partial\omega_{1x}$ and $\partial V_{1y}/\partial\omega_{1y}$ with P_{0y} as function of N_1.

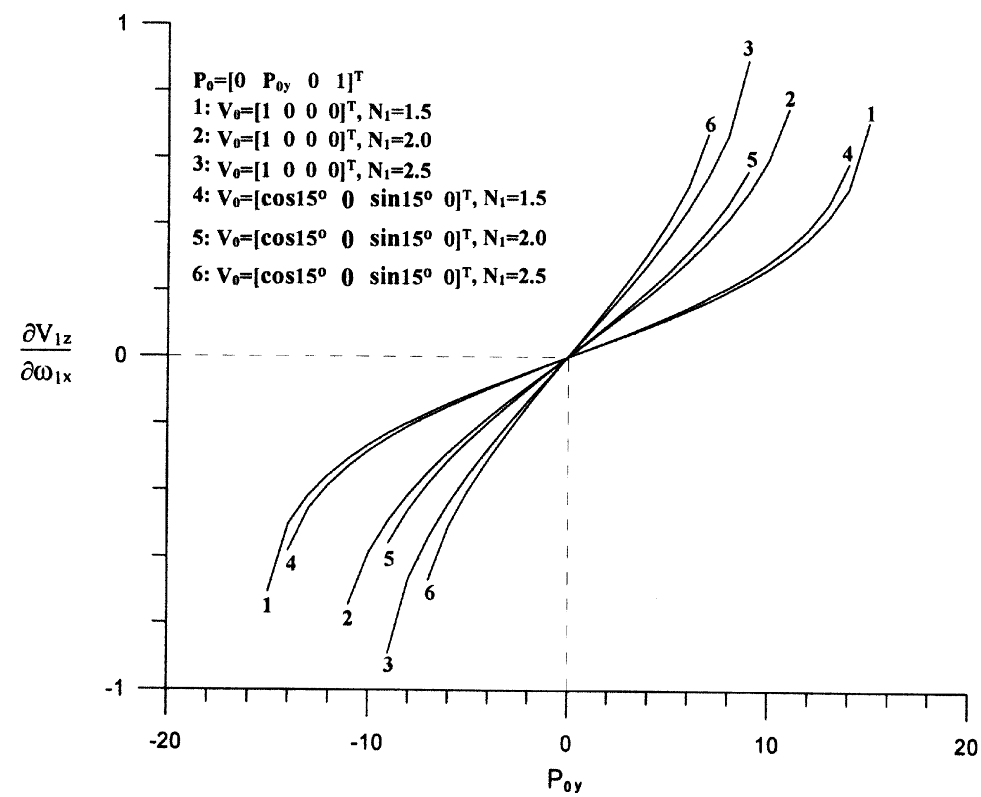

Fig. 8. Variation in $\partial V_{1z}/\partial\omega_{1x}$ with P_{0y} as function of N_1.

product of the sensitivity and error analysis matrices. The present results based on the analysis method presented above demonstrate the importance of performing an error analysis and subsequently applying the analysis method to the design process of the optical system.

5. CONCLUSION

In this paper, the author uses a homogeneous transformation matrix to define the position and orientation of a local coordinate frame for establishing each boundary surface and for tracing the light ray as it passes through a cylindrical boundary surface in accordance with Snell's law. Furthermore, the results of this paper present a direct and rapid analytical expression for the errors of a ray's light path as the ray passes through optical elements with cylindrical boundary surfaces. The proposed methodology considers two fundamental sources of error, namely translational and rotational errors at each boundary surface and the differential changes in the incident point position and unit directional vector of the refracted/reflected ray resulting from differential changes in the position and unit directional vector of the light source.

ACKNOWLEDGMENTS

The financial support provided to this study by the National Science Council of Taiwan under grant NSC96-2221-E-269-003 is gratefully acknowledged.

REFERENCES

1. Conrady, A.E., *Applied Optics and Optical Design, Part I*, Oxford Press (1929), reprinted by Dover Publications, Inc., New York, 1957.
2. Herzberger, M., *Modern Geometrical Optics*, New York, Interscience Publishers, 1958.
3. Hecht, E., *Optics*, 3rd Edition, Addison-Wesley Publishing Company, 1998.
4. Lin, P.D., "Analysis and Modeling of Optical Element and System," *J. Eng. Ind. Trans. ASME*, Vol. 116, pp. 101–107, 1994.
5. Forbes, G.W., "Optical system assessment for design: numerical ray tracing in the Gaussian pupil," *J. Opt. Soc. Am.*, Vol. A 5, pp. 1943–1956, 1988.
6. Forbes, G.W., "Accuracy doubling in the determination of the final ray configurations," *J. Opt. Soc. Am.*, Vol. A 6, pp. 1776–1783, 1989.
7. Stone, B.D., "Determination of initial ray configurations for asymmetric systems," *J. Opt. Soc. Am.*, Vol. A 14, pp. 3415–3429, 1997.
8. Lin, P.D. and Tsai, C.Y., "First-order gradients of skew rays of axis-symmetrical optical systems," *J. Opt. Soc. Am.*, Vol. A 24, pp. 776–784, 2007.
9. Liao, T.T. and Lin P.D., "Analysis of Optical Elements with Flat Boundary Surfaces," *Appl. Opt.*, Vol. 42, pp. 1191–1202, 2003.
10. Lin, P.D. and Lu, C.H., "Analysis and Design of Optical System by Use of Sensitivity Aanlysis of Skew Ray Tracing," *App. Opt.*, Vol. 43, pp. 796–807, 2004.
11. Lin, P.D. and Liao, T.T., "A New Model of Binocular Stereo Coordinate Measurement System Based on Skew Ray Tracing," *J. Dyn. Syst. Meas. Control. Trans. ASME*, Vol. 126, pp. 102–114, 2004.
12. Tsai, C.Y. and Lin, P.D., "Prism Design Based on Changes in Image Orientation," *App. Opt.*, Vol. 45, pp. 3951–2959, 2006.

13. Tlusty, J., "Techniques for Testing Accuracy of NC-Machine Tools," *12th MDTR-Conference*, 1972.
14. Ferreira, P.M. and Liu, R.C., "An Analytical Quadratic Model for the Geometric Error of a Machine Tool," *J. Manuf. Syst.*, Vol. 5, pp. 51–63, 1986.
15. Schultshik, R., "The Components of Volumetric Accuracy," *Ann. CIRP.*, Vol. 25, pp. 223–228, 1977.
16. Mahajan, V.N., *Optical Imaging and Aberrations*, Washington, SPIE Bellingham, 1998.
17. Paul, R.P., *Robot Manipulators-Mathematics, Programming Control*, Cambridge, MA, MIT press, 1982.

APPENDIX

$$\Delta H_i = \begin{bmatrix} \underline{H}_{i,1} & \underline{H}_{i,2} & \underline{H}_{i,3} & \underline{H}_{i,4} & \underline{H}_{i,5} & \underline{H}_{i,6} \end{bmatrix} \begin{bmatrix} \Delta P_{i-1} \\ \Delta V_{i-1} \end{bmatrix} + \begin{bmatrix} \underline{\underline{H}}_{i,1} & \underline{\underline{H}}_{i,2} & \underline{\underline{H}}_{i,3} & \underline{\underline{H}}_{i,4} & \underline{\underline{H}}_{i,5} & \underline{\underline{H}}_{i,6} \end{bmatrix} [\Delta e_i]. \quad \text{(A1)}$$

$$\underline{H}_{i,1} = \underline{H}_{i,2} = \underline{H}_{i,3} = 0 \quad \underline{H}_{i,4} = 2(I_{ix}G_{ix} + I_{iy}G_{iy}) \quad \underline{H}_{i,5} = 2(J_{ix}G_{ix} + J_{iy}G_{iy}) \quad \underline{H}_{i,6} = 2(K_{ix}G_{ix} + K_{iy}G_{iy})$$
$$\underline{\underline{H}}_{i,1} = \underline{\underline{H}}_{i,2} = \underline{\underline{H}}_{i,3} = \underline{\underline{H}}_{i,6} = 0 \quad \underline{\underline{H}}_{i,4} = 2G_{iy}G_{iz} \quad \underline{\underline{H}}_{i,5} = -2G_{ix}G_{iz}$$

$$\Delta D_i = \begin{bmatrix} \underline{D}_{i,1} & \underline{D}_{i,2} & \underline{D}_{i,3} & \underline{D}_{i,4} & \underline{D}_{i,5} & \underline{D}_{i,6} \end{bmatrix} \begin{bmatrix} \Delta P_{i-1} \\ \Delta V_{i-1} \end{bmatrix} + \begin{bmatrix} \underline{\underline{D}}_{i,1} & \underline{\underline{D}}_{i,2} & \underline{\underline{D}}_{i,3} & \underline{\underline{D}}_{i,4} & \underline{\underline{D}}_{i,5} & \underline{\underline{D}}_{i,6} \end{bmatrix} [\Delta e_i] \quad \text{(A2)}$$

$$\underline{D}_{i,1} = I_{ix}G_{ix} + I_{iy}G_{iy} \quad \underline{D}_{i,2} = J_{ix}G_{ix} + J_{iy}G_{iy} \quad \underline{D}_{i,3} = K_{ix}G_{ix} + K_{iy}G_{iy}$$
$$\underline{D}_{i,4} = I_{ix}(F_{ix} + t_{ix}) + I_{iy}(F_{iy} + t_{iy}) \quad \underline{D}_{i,5} = J_{ix}(F_{ix} + t_{ix}) + J_{iy}(F_{iy} + t_{iy})$$
$$\underline{D}_{i,6} = K_{ix}(F_{ix} + t_{ix}) + K_{iy}(F_{iy} + t_{iy}) \quad \underline{\underline{D}}_{i,1} = -G_{ix} \quad \underline{\underline{D}}_{i,2} = -G_{iy}$$
$$\underline{\underline{D}}_{i,3} = 0 \quad \underline{\underline{D}}_{i,6} = 0 \quad \underline{\underline{D}}_{i,4} = F_{iy}G_{iz} + F_{iz}G_{iy} + t_{iz}G_{iy} + t_{iy}G_{iz}$$
$$\underline{\underline{D}}_{i,5} = -(F_{ix}G_{iz} + F_{iz}G_{ix} + t_{iz}G_{ix} + t_{ix}G_{iz})$$

$$\Delta E_i = \begin{bmatrix} \underline{E}_{i,1} & \underline{E}_{i,2} & \underline{E}_{i,3} & \underline{E}_{i,4} & \underline{E}_{i,5} & \underline{E}_{i,6} \end{bmatrix} \begin{bmatrix} \Delta P_{i-1} \\ \Delta V_{i-1} \end{bmatrix} + \begin{bmatrix} \underline{\underline{E}}_{i,1} & \underline{\underline{E}}_{i,2} & \underline{\underline{E}}_{i,3} & \underline{\underline{E}}_{i,4} & \underline{\underline{E}}_{i,5} & \underline{\underline{E}}_{i,6} \end{bmatrix} [\Delta e_i] \quad \text{(A3)}$$

$$\underline{E}_{i,1} = 2I_{ix}(F_{ix} + t_{ix}) + 2I_{iy}(F_{iy} + t_{iy}) \quad \underline{E}_{i,2} = 2J_{ix}(F_{ix} + t_{ix}) + 2J_{iy}(F_{iy} + t_{iy})$$
$$\underline{E}_{i,3} = 2K_{ix}(F_{ix} + t_{ix}) + 2K_{iy}(F_{iy} + t_{iy}) \quad \underline{E}_{i,4} = \underline{E}_{i,5} = \underline{E}_{i,6} = 0 \quad \underline{\underline{E}}_{i,1} = -2(F_{ix} + t_{ix})$$
$$\underline{\underline{E}}_{i,2} = -2(F_{iy} + t_{iy}) \quad \underline{\underline{E}}_{i,3} = 0 \quad \underline{\underline{E}}_{i,6} = 0$$
$$\underline{\underline{E}}_{i,4} = 2(F_{iy}F_{iz} + F_{iy}t_{iz} + F_{iz}t_{iy} + t_{iy}t_{iz}) \quad \underline{\underline{E}}_{i,5} = -2(F_{ix}F_{iz} + F_{ix}t_{iz} + F_{iz}t_{ix} + t_{ix}t_{iz})$$

$$\Delta \lambda_i = \begin{bmatrix} \underline{\lambda}_{i,1} & \underline{\lambda}_{i,2} & \underline{\lambda}_{i,3} & \underline{\lambda}_{i,4} & \underline{\lambda}_{i,5} & \underline{\lambda}_{i,6} \end{bmatrix} \begin{bmatrix} \Delta P_{i-1} \\ \Delta V_{i-1} \end{bmatrix} + \begin{bmatrix} \underline{\underline{\lambda}}_{i,1} & \underline{\underline{\lambda}}_{i,2} & \underline{\underline{\lambda}}_{i,3} & \underline{\underline{\lambda}}_{i,4} & \underline{\underline{\lambda}}_{i,5} & \underline{\underline{\lambda}}_{i,6} \end{bmatrix} [\Delta e_i] \quad \text{(A4)}$$

$$\underline{\lambda}_{i,j} = \frac{1}{H_i}\left[-\underline{D}_{i,j} \pm \frac{1}{2}(D_i^2 - H_i \cdot E_i)^{\frac{-1}{2}}(2D_i \cdot \underline{D}_{i,j} - \underline{H}_{i,j} \cdot E_i - H_i \cdot \underline{E}_{i,j})\right] - \frac{\underline{H}_{i,j}}{H_i^2}\left[-D_i \pm (D_i^2 - H_i \cdot E_i)^{\frac{1}{2}}\right]$$

$(j = 1 \sim 6)$

$$\underline{\underline{\lambda}}_{i,j} = \frac{1}{H_i}\left[-\underline{\underline{D}}_{i,j} \pm \frac{1}{2}(D_i^2 - H_i \cdot E_i)^{\frac{-1}{2}}(2D_i \cdot \underline{\underline{D}}_{i,j} - \underline{H}_{i,j} \cdot E_i - H_i \cdot \underline{E}_{i,j})\right] - \frac{\underline{\underline{H}}_{i,j}}{H_i^2}\left[-D_i \pm (D_i^2 - H_i \cdot E_i)^{\frac{1}{2}}\right]$$

$(j = 1 \sim 6)$

$$\Delta\sigma_i = \left[\, \underline{\sigma}_{i,1} \quad \underline{\sigma}_{i,2} \quad \underline{\sigma}_{i,3} \quad \underline{\sigma}_{i,4} \quad \underline{\sigma}_{i,5} \quad \underline{\sigma}_{i,6} \,\right] \begin{bmatrix} \Delta P_{i-1} \\ \Delta V_{i-1} \end{bmatrix} + \left[\, \underline{\underline{\sigma}}_{i,1} \quad \underline{\underline{\sigma}}_{i,2} \quad \underline{\underline{\sigma}}_{i,3} \quad \underline{\underline{\sigma}}_{i,4} \quad \underline{\underline{\sigma}}_{i,5} \quad \underline{\underline{\sigma}}_{i,6} \,\right] [\Delta e_i] \qquad \text{(A5)}$$

$$\underline{\sigma}_{i,1} = I_{ix} + G_{ix}\underline{\lambda}_{i,1} \quad \underline{\sigma}_{i,2} = J_{ix} + G_{ix}\underline{\lambda}_{i,2} \quad \underline{\sigma}_{i,3} = K_{ix} + G_{ix}\underline{\lambda}_{i,3} \quad \underline{\sigma}_{i,4} = I_{ix}\lambda_i + G_{ix}\underline{\lambda}_{i,4}$$
$$\underline{\sigma}_{i,5} = J_{ix}\lambda_i + G_{ix}\underline{\lambda}_{i,5} \quad \underline{\sigma}_{i,6} = K_{ix}\lambda_i + G_{ix}\underline{\lambda}_{i,6} \quad \underline{\underline{\sigma}}_{i,1} = -1 + G_{ix}\underline{\underline{\lambda}}_{i,1} \quad \underline{\underline{\sigma}}_{i,2} = G_{ix}\underline{\underline{\lambda}}_{i,2} \quad \underline{\underline{\sigma}}_{i,3} = G_{ix}\underline{\underline{\lambda}}_{i,3}$$
$$\underline{\underline{\sigma}}_{i,4} = G_{ix}\underline{\underline{\lambda}}_{i,4} \quad \underline{\underline{\sigma}}_{i,5} = -F_{iz} - \lambda_i G_{iz} - t_{iz} + G_{ix}\underline{\underline{\lambda}}_{i,5} \quad \underline{\underline{\sigma}}_{i,6} = F_{iy} + \lambda_i G_{iy} + t_{iy} + G_{ix}\underline{\underline{\lambda}}_{i,6}$$

$$\Delta\rho_i = \left[\, \underline{\rho}_{i,1} \quad \underline{\rho}_{i,2} \quad \underline{\rho}_{i,3} \quad \underline{\rho}_{i,4} \quad \underline{\rho}_{i,5} \quad \underline{\rho}_{i,6} \,\right] \begin{bmatrix} \Delta P_{i-1} \\ \Delta V_{i-1} \end{bmatrix} + \left[\, \underline{\underline{\rho}}_{i,1} \quad \underline{\underline{\rho}}_{i,2} \quad \underline{\underline{\rho}}_{i,3} \quad \underline{\underline{\rho}}_{i,4} \quad \underline{\underline{\rho}}_{i,5} \quad \underline{\underline{\rho}}_{i,6} \,\right] [\Delta e_i] \qquad \text{(A6)}$$

$$\underline{\rho}_{i,1} = I_{iy} + G_{iy}\underline{\lambda}_{i,1} \quad \underline{\rho}_{i,2} = J_{iy} + G_{iy}\underline{\lambda}_{i,2} \quad \underline{\rho}_{i,3} = K_{iy} + G_{iy}\underline{\lambda}_{i,3} \quad \underline{\rho}_{i,4} = I_{iy}\lambda_i + G_{iy}\underline{\lambda}_{i,4}$$
$$\underline{\rho}_{i,5} = J_{iy}\lambda_i + G_{iy}\underline{\lambda}_{i,5} \quad \underline{\rho}_{i,6} = K_{iy}\lambda_i + G_{iy}\underline{\lambda}_{i,6} \quad \underline{\underline{\rho}}_{i,1} = G_{iy}\underline{\underline{\lambda}}_{i,1} \quad \underline{\underline{\rho}}_{i,2} = G_{iy}\underline{\underline{\lambda}}_{i,2} - 1 \quad \underline{\underline{\rho}}_{i,3} = G_{iy}\underline{\underline{\lambda}}_{i,3}$$
$$\underline{\underline{\rho}}_{i,4} = G_{iy}\underline{\underline{\lambda}}_{i,4} + (F_{iz} + t_{iz}) + \lambda_i G_{iz} \quad \underline{\underline{\rho}}_{i,5} = G_{iy}\underline{\underline{\lambda}}_{i,5} \quad \underline{\underline{\rho}}_{i,6} = G_{iy}\underline{\underline{\lambda}}_{i,6} - (F_{ix} + t_{ix}) - \lambda_i G_{ix}$$

$$\Delta\alpha_i = \left[\, \underline{\alpha}_{i,1} \quad \underline{\alpha}_{i,2} \quad \underline{\alpha}_{i,3} \quad \underline{\alpha}_{i,4} \quad \underline{\alpha}_{i,5} \quad \underline{\alpha}_{i,6} \,\right] \begin{bmatrix} \Delta P_{i-1} \\ \Delta V_{i-1} \end{bmatrix} + \left[\, \underline{\underline{\alpha}}_{i,1} \quad \underline{\underline{\alpha}}_{i,2} \quad \underline{\underline{\alpha}}_{i,3} \quad \underline{\underline{\alpha}}_{i,4} \quad \underline{\underline{\alpha}}_{i,5} \quad \underline{\underline{\alpha}}_{i,6} \quad \underline{\underline{\alpha}}_{i,7} \,\right] [\Delta e_i] \qquad \text{(A7)}$$

$$\underline{\alpha}_{i,j} = \frac{\sigma_i \underline{\rho}_{i,j} - \rho_i \underline{\sigma}_{i,j}}{\sigma_i^2 + \rho_i^2} \;\; (j = 1 \sim 6) \qquad \underline{\underline{\alpha}}_{i,j} = \frac{\sigma_i \underline{\underline{\rho}}_{i,j} - \rho_i \underline{\underline{\sigma}}_{i,j}}{\sigma_i^2 + \rho_i^2} \;\; (j = 1 \sim 6)$$

$$\begin{bmatrix} \Delta n_{ix} \\ \Delta n_{iy} \\ \Delta n_{iz} \end{bmatrix} = \begin{bmatrix} \underline{n}_{ix,1} & \underline{n}_{ix,2} & \underline{n}_{ix,3} & \underline{n}_{ix,4} & \underline{n}_{ix,5} & \underline{n}_{ix,6} \\ \underline{n}_{iy,1} & \underline{n}_{iy,2} & \underline{n}_{iy,3} & \underline{n}_{iy,4} & \underline{n}_{iy,5} & \underline{n}_{iy,6} \\ \underline{n}_{iz,1} & \underline{n}_{iz,2} & \underline{n}_{iz,3} & \underline{n}_{iz,4} & \underline{n}_{iz,5} & \underline{n}_{iz,6} \end{bmatrix} \begin{bmatrix} \Delta P_{i-1} \\ \Delta V_{i-1} \end{bmatrix} + \begin{bmatrix} \underline{\underline{n}}_{ix,1} & \underline{\underline{n}}_{ix,2} & \underline{\underline{n}}_{ix,3} & \underline{\underline{n}}_{ix,4} & \underline{\underline{n}}_{ix,5} & \underline{\underline{n}}_{ix,6} \\ \underline{\underline{n}}_{iy,1} & \underline{\underline{n}}_{iy,2} & \underline{\underline{n}}_{iy,3} & \underline{\underline{n}}_{iy,4} & \underline{\underline{n}}_{iy,5} & \underline{\underline{n}}_{iy,6} \\ \underline{\underline{n}}_{iz,1} & \underline{\underline{n}}_{iz,2} & \underline{\underline{n}}_{iz,3} & \underline{\underline{n}}_{iz,4} & \underline{\underline{n}}_{iz,5} & \underline{\underline{n}}_{iz,6} \end{bmatrix} [\Delta e_i] \qquad \text{(A8)}$$

$$\underline{n}_{ix,j} = s_i(I_{iy}C\alpha_i - I_{ix}S\alpha_i)\underline{\alpha}_{i,j} \quad \underline{n}_{iy,j} = s_i(J_{iy}C\alpha_i - J_{ix}S\alpha_i)\underline{\alpha}_{i,j} \quad \underline{n}_{iz,j} = s_i(K_{iy}C\alpha_i - K_{ix}S\alpha_i)\underline{\alpha}_{i,j} \quad (j = 1 \sim 6)$$
$$\underline{\underline{n}}_{ix,j} = s_i(I_{iy}C\alpha_i - I_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,j} \quad \underline{\underline{n}}_{iy,j} = s_i(J_{iy}C\alpha_i - J_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,j} \quad \underline{\underline{n}}_{iz,j} = s_i(K_{iy}C\alpha_i - K_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,j} \quad (j = 1 \sim 3)$$
$$\underline{\underline{n}}_{ix,4} = s_i(I_{iy}C\alpha_i - I_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,4} + I_{iz}S\alpha_i \quad \underline{\underline{n}}_{ix,5} = s_i(I_{iy}C\alpha_i - I_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,5} - I_{iz}C\alpha_i$$
$$\underline{\underline{n}}_{ix,6} = s_i(I_{iy}C\alpha_i - I_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,6} + I_{iy}C\alpha_i - I_{ix}S\alpha_i \quad \underline{\underline{n}}_{iy,4} = s_i(J_{iy}C\alpha_i - J_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,4} + J_{iz}S\alpha_i$$
$$\underline{\underline{n}}_{iy,5} = s_i(J_{iy}C\alpha_i - J_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,5} - J_{iz}C\alpha_i \quad \underline{\underline{n}}_{iy,6} = s_i(J_{iy}C\alpha_i - J_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,6} + J_{iy}C\alpha_i - J_{ix}S\alpha_i$$
$$\underline{\underline{n}}_{iz,4} = s_i(K_{iy}C\alpha_i - K_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,4} + K_{iz}S\alpha_i \quad \underline{\underline{n}}_{iz,5} = s_i(K_{iy}C\alpha_i - K_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,5} - K_{iz}C\alpha_i$$
$$\underline{\underline{n}}_{iy,6} = s_i(J_{iy}C\alpha_i - J_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,6} + J_{iy}C\alpha_i - J_{ix}S\alpha_i$$

$$\Delta C\theta_i = \left[\, \underline{C\theta}_{i,1} \quad \underline{C\theta}_{i,2} \quad \underline{C\theta}_{i,3} \quad \underline{C\theta}_{i,4} \quad \underline{C\theta}_{i,5} \quad \underline{C\theta}_{i,6} \,\right] \begin{bmatrix} \Delta P_{i-1} \\ \Delta V_{i-1} \end{bmatrix} + \left[\, \underline{\underline{C\theta}}_{i,1} \quad \underline{\underline{C\theta}}_{i,2} \quad \underline{\underline{C\theta}}_{i,3} \quad \underline{\underline{C\theta}}_{i,4} \quad \underline{\underline{C\theta}}_{i,5} \quad \underline{\underline{C\theta}}_{i,6} \,\right] [\Delta e_i] \qquad \text{(A9)}$$

$$
\begin{aligned}
&\underline{C\theta}_{i,j} = -s_i(G_{iy}C\alpha_i - G_{ix}S\alpha_i)\underline{\alpha}_{i,j} \qquad \underline{\underline{C\theta}}_{i,j} = -s_i(G_{iy}C\alpha_i - G_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,j} \qquad (j=1\sim 3)\\
&\underline{C\theta}_{i,4} = -s_i[(G_{iy}C\alpha_i - G_{ix}S\alpha_i)\underline{\alpha}_{i,4} + I_{ix}C\alpha_i + I_{iy}S\alpha_i]\\
&\underline{C\theta}_{i,5} = -s_i[(G_{iy}C\alpha_i - G_{ix}S\alpha_i)\underline{\alpha}_{i,5} + J_{ix}C\alpha_i + J_{iy}S\alpha_i]\\
&\underline{C\theta}_{i,6} = -s_i[(G_{iy}C\alpha_i - G_{ix}S\alpha_i)\underline{\alpha}_{i,4} + K_{ix}C\alpha_i + K_{iy}S\alpha_i] \qquad \underline{\underline{C\theta}}_{i,4} = -s_i[(G_{iy}C\alpha_i - G_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,4} + G_{iz}S\alpha_i]\\
&\underline{\underline{C\theta}}_{i,5} = -s_i[(G_{iy}C\alpha_i - G_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,5} - G_{iz}C\alpha_i] \qquad \underline{\underline{C\theta}}_{i,6} = -s_i[(G_{iy}C\alpha_i - G_{ix}S\alpha_i)\underline{\underline{\alpha}}_{i,6} + G_{iy}C\alpha_i - G_{ix}S\alpha_i]
\end{aligned}
$$

The differential changes of a light ray refracted (reflected) by cylindrical boundary surface has the form

$$
\begin{bmatrix} \Delta P_i \\ \Delta V_i \end{bmatrix} = \underline{\underline{M}}_i[e_i] + \underline{M}_i \begin{bmatrix} \Delta P_{i-1} \\ \Delta V_{i-1} \end{bmatrix} \quad \underline{\underline{M}}_i = [\underline{\underline{m}}_{ijk}] \quad \underline{M}_i = [\underline{m}_{ijk}] \quad (j=1\sim 6,\ k=1\sim 6) \tag{A10}
$$

$$
\begin{bmatrix} \Delta P_i \\ \Delta \underline{V}_i \end{bmatrix} = \underline{\underline{M}}_i[e_i] + \underline{M}_i \begin{bmatrix} \Delta P_{i-1} \\ \Delta V_{i-1} \end{bmatrix} \quad \underline{\underline{M}}_i = [\underline{\underline{m}}_{ijk}] \quad \underline{M}_i = [\underline{m}_{ijk}] \quad (j=1\sim 6,\ k=1\sim 6) \tag{A11}
$$

For the case of a refracted light ray:

$$
\underline{m}_{i1k} = \delta_{1k} + \lambda_i\delta_{4k} + V_{i-1x}\underline{\lambda}_{i,k} \qquad \underline{m}_{i2k} = \delta_{2k} + \lambda_i\delta_{5k} + V_{i-1y}\underline{\lambda}_{i,k} \qquad \underline{m}_{i3k} = \delta_{3k} + \lambda_i\delta_{6k} + V_{i-1z}\underline{\lambda}_{i,k}
$$

$$
\underline{m}_{i4k} = \left(\frac{-n_{ix}N_i^2C\theta_i}{\sqrt{1-N_i^2+(N_iC\theta_i)^2}} + N_in_{ix} \right)\underline{C\theta}_{i,k} + N_i + (-\sqrt{1-N_i^2+(N_iC\theta_i)^2} + N_iC\theta_i)\underline{n}_{ix,k}
$$

$$
\underline{m}_{i5k} = \left(\frac{-n_{iy}N_i^2C\theta_i}{\sqrt{1-N_i^2+(N_iC\theta_i)^2}} + N_in_{iy} \right)\underline{C\theta}_{i,k} + N_i + (-\sqrt{1-N_i^2+(N_iC\theta_i)^2} + N_iC\theta_i)\underline{n}_{iy,k}
$$

$$
\underline{m}_{i6k} = \left(\frac{-n_{iz}N_i^2C\theta_i}{\sqrt{1-N_i^2+(N_iC\theta_i)^2}} + N_in_{iz} \right)\underline{C\theta}_{i,k} + N_i + (-\sqrt{1-N_i^2+(N_iC\theta_i)^2} + N_iC\theta_i)\underline{n}_{iz,k}
$$

$$
\underline{\underline{m}}_{i1k} = V_{i-1x}\underline{\underline{\lambda}}_{i,k} \qquad \underline{\underline{m}}_{i2k} = V_{i-1y}\underline{\underline{\lambda}}_{i,k} \qquad \underline{\underline{m}}_{i3k} = V_{i-1z}\underline{\underline{\lambda}}_{i,k}
$$

$$
\underline{\underline{m}}_{i4k} = \left(\frac{-n_{ix}N_i^2C\theta_i}{\sqrt{1-N_i^2+(N_iC\theta_i)^2}} + N_in_{ix} \right)\underline{\underline{C\theta}}_{i,k} - (\sqrt{1-N_i^2+(N_iC\theta_i)^2} - N_iC\theta_i)\underline{\underline{n}}_{ix,k}
$$

$$
\underline{\underline{m}}_{i5k} = \left(\frac{-n_{iy}N_i^2C\theta_i}{\sqrt{1-N_i^2+(N_iC\theta_i)^2}} + N_in_{iy} \right)\underline{\underline{C\theta}}_{i,k} - (\sqrt{1-N_i^2+(N_iC\theta_i)^2} - N_iC\theta_i)\underline{\underline{n}}_{iy,k}
$$

$$\underline{\underline{m}}_{i6k} = \left(\frac{-n_{iz}N_i^2 C\theta_i}{\sqrt{1-N_i^2+(N_iC\theta_i)^2}} + N_i n_{iz} \right) \underline{\underline{C\theta}}_{i,k} - (\sqrt{1-N_i^2+(N_iC\theta_i)^2} - N_iC\theta_i)\underline{\underline{n}}_{iz,k}$$

For the case of a reflected light ray:

$$\underline{m}_{i1k} = \delta_{1k} + \lambda_i\delta_{4k} + V_{i-1x}\underline{\lambda}_{i,k} \quad \underline{m}_{i2k} = \delta_{2k} + \lambda_i\delta_{5k} + V_{i-1y}\underline{\lambda}_{i,k} \quad \underline{m}_{i3k} = \delta_{3k} + \lambda_i\delta_{6k} + V_{i-1z}\underline{\lambda}_{i,k}$$

$$\underline{m}_{i4k} = \delta_{4k} + 2n_{ix}\underline{C\theta}_{i,k} + 2C\theta_i\underline{n}_{ix,k} \quad \underline{m}_{i5k} = \delta_{5k} + 2n_{iy}\underline{C\theta}_{i,k} + 2C\theta_i\underline{n}_{iy,k} \quad \underline{m}_{i6k} = \delta_{6k} + 2n_{iz}\underline{C\theta}_{i,k} + 2C\theta_i\underline{n}_{ix,k}$$

$$\underline{\underline{m}}_{i1k} = V_{i-1x}\underline{\underline{\lambda}}_{i,k} \quad \underline{\underline{m}}_{i2k} = V_{i-1y}\underline{\underline{\lambda}}_{i,k} \quad \underline{\underline{m}}_{i3k} = V_{i-1z}\underline{\underline{\lambda}}_{i,k} \quad \underline{\underline{m}}_{i4k} = 2n_{ix}\underline{\underline{C\theta}}_{i,k} + 2C\theta_i\underline{\underline{n}}_{ix,k}$$

$$\underline{\underline{m}}_{i5k} = 2n_{iy}\underline{\underline{C\theta}}_{i,k} + 2C\theta_i\underline{\underline{n}}_{iy,k} \quad \underline{\underline{m}}_{i6k} = 2n_{iz}\underline{\underline{C\theta}}_{i,k} + 2C\theta_i\underline{\underline{n}}_{ix,k}$$

NOMENCLATURE

r_i	i-th boundary surface with unit normal n_i.
${}^kA_g$	configuration of frame $(xyz)_g$ with respect to frame $(xyz)_k$.
$(xyz)_0$	world coordinate frame.
$(xyz)_i$	ideal coordinate frame embedded in i-th boundary surface.
$(xyz)_a$	actual coordinate frame embedded in i-th boundary surface.
P_i	incident point on i-th boundary surface.
V_i	unit directional vector of light ray following reflection/refraction at r_i.
θ_i and $\underline{\theta}_i$	incident angle and refracted angle, respectively.
N_i	$N_i = \xi_{i-1}/\xi_i$, where ξ_i is refractive index of medium i with respect to vacuum.
ΔP_i	differential change in incident point position P_i.
ΔV_i	differential change in unit directional vector V_i.
$\underline{M}_i$	6×6 sensitivity matrix defined by $[\Delta P_i \quad \Delta V_i]^T = \underline{M}_i[\Delta P_{i-1} \quad \Delta V_{i-1}]^T$ when a light ray passes through the i-th boundary surface r_i.
$\underline{\underline{M}}_i$	6×6 error matrix defined by $[\Delta P_i \quad \Delta V_i]^T = \underline{\underline{M}}_i[\Delta d_{ix} \quad \Delta d_{iy} \quad \Delta d_{iz} \quad \Delta\omega_{ix} \quad \Delta\omega_{iy} \quad \Delta\omega_{iz}]^T$, when light ray passes through the i-th boundary surface r_i.
M_i	6×6 error analysis matrix based on coordinate frame $(xyz)_i$ of the i-th boundary surface.