GEOMETRY AND SIMULATION OF THE GENERATION OF CYLINDRICAL GEARS BY AN IMAGINARY DISC CUTTER

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ABSTRACT

This paper proposed a method of representing the geometry of an imaginary disc cutter in parameter form. The undercutting condition of the imaginary disc cutter is studied, and the mathematical model of the generated gears is developed by the undercutting condition of the cutter. Through a mathematical model of the generation process, the vector equations of generated gears are established. According to the proposed method, a planetary gear mechanism and a pair of gear pump with smaller numbers of teeth are illustrated. A cutting simulation process is presented for machining the proposed gear pairs. Stress analysis for the proposed gear mechanism is performed. Finally, the proposed method is applied to determine singular points of the proposed disc cutter.

Keywords: gears, disc cutter.

GÉOMÉTRIE ET DE SIMULATION DE LA GÉNÉRATION DES ENGRENAGES CYLINDRIQUES D'UN IMAGINAIRE DISC CUTTER

RÉSUMÉ

Ce document propose une méthode de représentation de la géométrie d'un disque de coupe dans l'imaginaire sous forme de paramètre. La sous-condition de l'imaginaire du disque de coupe est étudié, et le modèle mathématique de la génération d'engins est développé par la sous-condition de la fraise. Grâce à un modèle mathématique du processus de génération, le vecteur des équations générées engins sont mis en place. Selon la méthode proposée, un mécanisme d'engrenage planétaire et d'une paire de pompe avec de plus petits nombres de dents sont illustrées. Un processus de simulation de coupe pour l'usinage est présenté le projet de train paires. Analyse de la proposition de mécanisme de train est effectué. Enfin, la méthode proposée est appliquée à déterminer les points singuliers de la proposition de disque de coupe.

1. INTRODUCTION

Spur gears are widely used in the industries for parallel axis transmission, and may be generated by a hob, a sharper, and a rack cutter. Therefore, many researchers and gear companies contribute valuable support to the design and manufacture of the spur gear. Conventional spur gears are involute curves, and many studies are conducted on involute gears and almost spur gears generated by imaginary rack cutters. Yang [1-4] recently presented a design in which, a rack cutter with ring involute teeth was used to generate a pinion and a gear. Also, a direct gear design for spur and helical involute gears was presented by Alexander and Roderick [5], and a rack cutter with asymmetric involute teeth for generating spur gear was presented by Alexander [6]. In addition to, a rack cutter can also be used to generate elliptical gears [13]. Typically, the normal section of the rack cutter-involute generating portion is a straight line; however, an imaginary disc cutter with both a straight line and a circular arc curve is presented in this work. This proposed cutter can be used to generate a gear pump with a few teeth and a planetary gear mechanism.

A pair of spur gears with a few teeth can be used for a pump mechanism. Chen [7] presented a gear pump with circular arc teeth, and proposed an inverse concept for determining a rack cutter. In another study, Ishibashi [8] proposed a mathematical model of the spur gear with a few teeth. Also, Mimmi and Pennacchi [9] presented an analytical model of a particular type of positive displacement blower. Wherein a cutter of three lobes with epitrochoid and involute arc was used to obtain the geometrical model of the pump mechanism. Reference [10] shows the shape cutter of an involute curve that is used to generate a helical gear with a few teeth. References [11,12] show the use of a rack cutter with double circular-arc tooth to generate a gear. A tooth profile of elliptical gears with circular-arc teeth was presented by Chan et al. [13]. In his work, a rack cutter with circular-arc was used to generate an elliptical gear, but it was not adaptable for generating a ring gear. However, very few studies have reported using an imaginary disc cutter consisting of a straight-line and a circular arc curve to generate a gear pump. Nevertheless, such a disc cutter is potentially a shape cutter that could generate a nonstandard gear. Moreover, the cutter is adaptable for generating a ring gear.

Therefore, this paper proposed a disc cutter consisting of a straight-line and circular arc curve to generate any gear pairs. Based on gear theory, the proposed cutter becomes a generating curve, which in turn becomes the envelope to the parameter family of the generating curves. During the generation process of a generating curve, singular points may exist on the generated curve. However, if such singular points occur, the load capacity near the fillets of the tooth will be decreased. Singular points of spur gears have been studied by Mabie and Reinholtz [14] using geometric relationships.

In order to prevent a singular point, the undercutting condition of the designed cutter must be properly selected. To determine the undercutting condition of the proposed cutter, a representation of the geometric meaning of meshing equations is applied to the design tool for the gears. The results of this research may be helpful to the design of the proposed disc cutter.

An analytical expression of a disc cutter with a straight line and arc curve teeth is also presented. A disc cutter with a straight line and arc curve teeth is shown in Fig. 1. According to gear theory [15], a gear and a pinion or a sun gear and a ring gear can be obtained as an envelope to the family of an imaginary rack cutter in different positions when the pinion and the gear or the sun gear and the ring gear rotate for a cycle. Based on the obtained mathematical models and running the *SolidWorks* software on the *Windows XP*, three dimensional geometric models is generated. The model is then transferred to the FE package *ANSYS Workbench* [16]

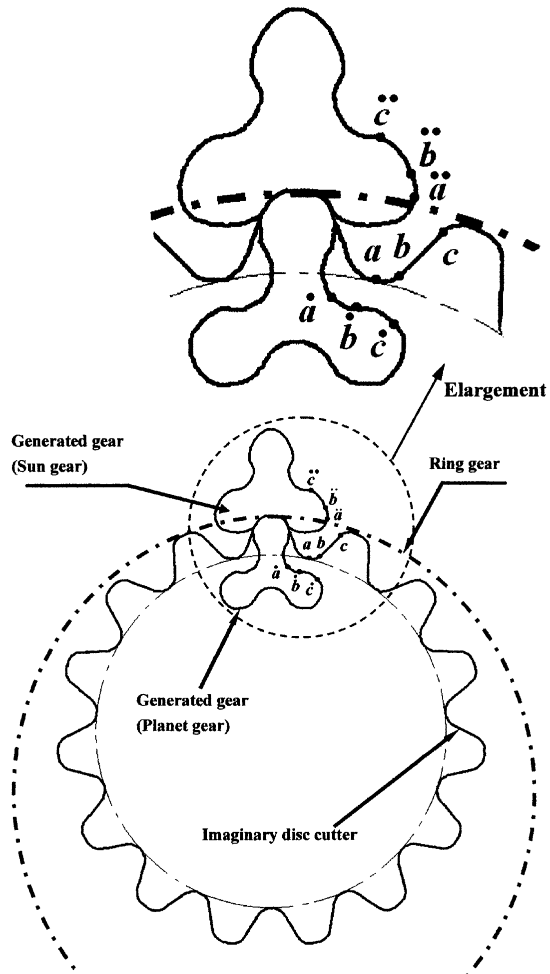

Fig. 1. The disc cutter composition with straight-line segment $\overline{bc}$ and circular arcs $\overline{ab}$ ($\theta_x = \dfrac{\pi}{60\eta}$).

for stress analysis. In order to determine the maximum contact stress for the planetary gear mechanism the parameter of the proposed disc cutter can be selected.

2. GEOMETRY OF DISC CUTTER WITH STRAIGHT LINE AND ARC CURVE

The proposed imaginary disc cutter can be used to obtain two generated gears: One is sun gear; the other is planet gear. The two gears can be generated by the cutters which simultaneously perform rotational motion with the two gears. As shown in Fig. 1, the circular-arc curve $\overline{ab}$ and the straight-line segment $\overline{bc}$ of the imaginary disc cutter are illuminated. The working tooth curve side of the sun gear $\overline{\ddot{b}\ddot{c}}$ is generated by the straight-line segment $\overline{bc}$ of the imaginary disc cutter. Similarly, the working tooth curve side of the planet gear $\overline{\dot{b}\dot{c}}$ is also generated by the straight-line segment $\overline{bc}$ of the imaginary disc cutter. The circular-arc curve $\overline{ab}$ of the imaginary disc cutter is used to generate the tip fillet of a sun gear $\overline{\ddot{a}\ddot{b}}$ and the root fillet of the planet gear $\overline{\dot{a}\dot{b}}$. Similarly, another circular-arc curve $\overline{cd}$ of the imaginary disc cutter in Fig. 3a is applied to determine a root fillet of the sun gear and a tip fillet of planet gears. The disc cutter can be divided into two imaginary disc cutters. One is Σ_t for generating the sun gear, as shown in Fig. 2a. The other is Σ_c for generating the planet gear, as shown in Fig. 2b. Using the imaginary disc cutters Σ_t and Σ_c, a gear pump with a few teeth can be generated. To obtain a planetary gear mechanism, only the imaginary disc cutter Σ_t will be used, as will be further described in the following sections. The mathematical model of the cutter Σ_t can be demonstrated. The cutter Σ_c can also be determined using a similar method.

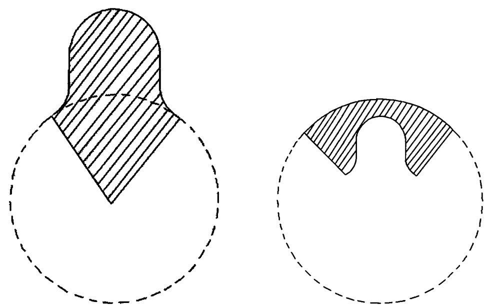

Fig. 2a. The imaginary disc cutter Σ_t is used to generate the sun gear. 2b. The imaginary disc cutter Σ_c is used to generate the planet gear.

To obtain a parameter expression of the proposed disc cutter, the design of a disc cutter with one tooth is discussed here. The shape of such a disc cutter consists of a straight line and circular arc curve, as shown in Fig. 3a. The height of the tooth is h. The taper angle of the tooth, representing the pressure angle of a generated gear is α. The coordinate system $S_c(O_c,\ x_c,\ y_c,\ z_c)$ is fixed in a tooth. The axis z_c, as shown in Fig. 3a, is determined by a right coordinate system. In order to generate the complete profile of the cutter curves, the tooth of the proposed disc cutter is repeated for r_θ with respect to the coordinate system $S_1(O_1,\ x_1,\ y_1,\ z_1)$, as shown in Fig. 3a. Here, r_θ is equal to $2\pi i/n_1$ (i=1,2,3,…n_1). The number of teeth of the proposed disc cutter is n_1.

2-1 Regions $\overline{ab}$ and $\overline{ih}$

As shown in Figs. 3a and 3c, regions $\overline{ab}$ and $\overline{ih}$ are used to generate the tip fillet of a sun gear and the root fillet of a planet gear, and θ_t represents a design parameter of the proposed cutter. Based on Figs. 3a and 3c, the equation for the regions $\overline{ab}$ and $\overline{ih}$, represented in the coordinate system S_1, can be written as:

$$\mathbf{R}_1^{ab}=\begin{bmatrix} x_1^{ab}\\ y_1^{ab}\\ z_1^{ab}\\ 1\end{bmatrix}=\begin{bmatrix} ((r+r_{c1})\sin(\theta_p+\theta_x)-r_{c1}\cos(\theta_t))\cos r_\theta \\ +((r+r_{c1})\cos(\theta_p+\theta_x)-r_{c1}\sin(\theta_t))\sin r_\theta \\ \\ -((r+r_{c1})\sin(\theta_p+\theta_x)-r_{c1}\cos(\theta_t))\sin r_\theta \\ +((r+r_{c1})\cos(\theta_p+\theta_x)-r_{c1}\sin(\theta_t))\cos r_\theta \\ \\ 0 \\ \\ 1\end{bmatrix},0<\theta_t<(\theta_{c3}+\theta_{c2}), \quad (1)$$

and

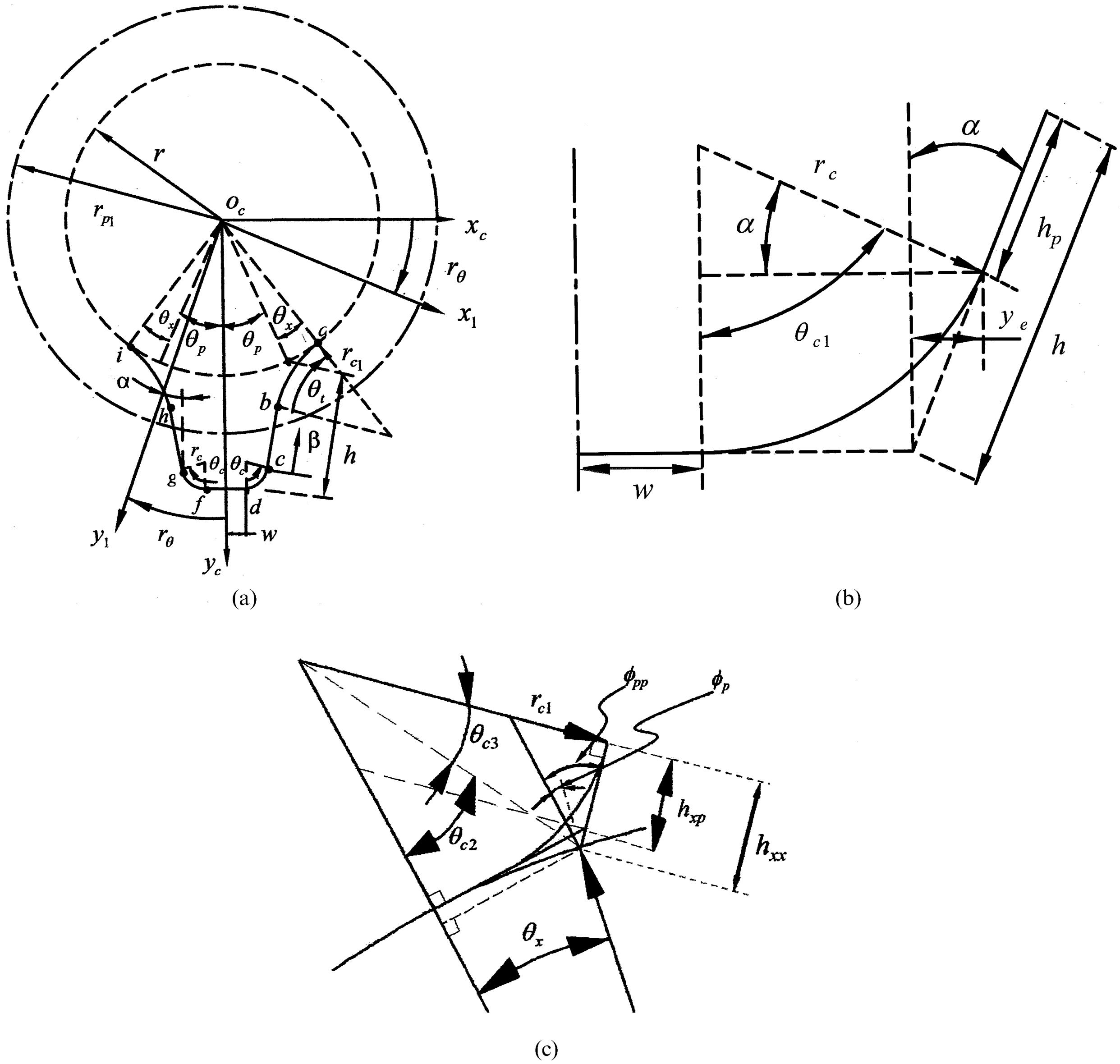

Fig. 3a. The proposed disc cutter design. 3b. The detail symbols on region $\overline{cd}$. 3c. The detail symbols on region $\overline{ab}$.

$$\mathbf{R}_1^{ih} = \begin{bmatrix} x_1^{ih} \\ y_1^{ih} \\ z_1^{ih} \\ 1 \end{bmatrix} = \begin{bmatrix} -((r+r_{c1})\sin(\theta_p+\theta_x) - r_{c1}\cos(\theta_t))\cos r_\theta \\ +((r+r_{c1})\cos(\theta_p+\theta_x) - r_{c1}\sin(\theta_t))\sin r_\theta \\ \\ ((r+r_{c1})\sin(\theta_p+\theta_x) - r_{c1}\cos(\theta_t))\sin r_\theta \\ +((r+r_{c1})\cos(\theta_p+\theta_x) - r_{c1}\sin(\theta_t))\cos r_\theta \\ \\ 0 \\ \\ 1 \end{bmatrix}, 0<\theta_t<(\theta_{c3}+\theta_{c2}). \quad (2)$$

where

$$\theta_{c3} = \tan^{-1}(\frac{h_{xx}}{r_{c1}})$$

$$h_{xx} = h_{xp} + (\frac{r - r\cos\theta_x}{\cos\phi_{pp}})$$

$$h_{xp} = (r - r\cos\theta_x)\tan\phi_{pp} + r\sin\theta_x$$

$$r_{c1} = \frac{2r^2(1 - \cos\theta_x) - h^2{}_{xx})}{2r(\cos\theta_x - 1)}$$

$$\phi_{pp} = \phi_p + \theta_p + \alpha$$

$$\phi_p = \tan^{-1}(\frac{r\tan\theta_x - r\sin\theta_x}{r - r\cos\theta_x})$$

$$\theta_x = \frac{10\pi}{180\eta}$$

$$\theta_p = (\frac{\pi}{n_1} - \theta_x)/\eta$$

$$\theta_{c2} = \tan^{-1}(\frac{r\sin\theta_x}{r + r_{c1} - r\cos\theta_x})$$

where only parameters r, r_{c1}, n_1, θ_x, θ_p, and η are given values. The other parameters are determined by analytical geometry of the proposed disc cutter. The symbol η is the modified coefficient of the tooth profile, and is generally used to generate the backlash between the gear meshing. The parameter η can't equal to zero. The basic radius of the proposed disc cutter is r. The radius of the root fillet curve of the disc cutter is r_{c1}.

2-2 Regions $\overline{bc}$ and $\overline{hg}$

Straight-line regions $\overline{bc}$ and $\overline{hg}$ on the disc cutter is used to generate the side of the sun and the planet gear working tooth curve. β represents the curvilinear parameter that determines the coordinates of any points on the working tooth curve. The position vector of regions $\overline{bc}$ and $\overline{hg}$ is represented in the coordinate system S_1 as follows:

$$\boldsymbol{R}_1^{bc} = \begin{bmatrix} x_1^{bc} \\ y_1^{bc} \\ z_1^{bc} \\ 1 \end{bmatrix} = \begin{bmatrix} (r\sin\theta_p - (h_p - \beta)\sin\alpha)\cos r_\theta + (r\cos\theta_p + (h_p - \beta)\cos\alpha)\sin r_\theta \\ -(r\sin\theta_p - (h_p - \beta)\sin\alpha)\sin r_\theta + (r\cos\theta_p + (h_p - \beta)\cos\alpha)\cos r_\theta \\ 0 \\ 1 \end{bmatrix}, \tag{3}$$

$$0 < \beta < h_p - h_{xx},$$

and

$$\boldsymbol{R}^{hg} = \begin{bmatrix} x_1^{hg} \\ y_1^{hg} \\ z_1^{hg} \\ 1 \end{bmatrix} = \begin{bmatrix} -(r\sin\theta_p - (h_p - \beta)\sin\alpha)\cos r_\theta + (r\cos\theta_p + (h_p - \beta)\cos\alpha)\sin r_\theta \\ (r\sin\theta_p - (h_p - \beta)\sin\alpha)\sin r_\theta + (r\cos\theta_p + (h_p - \beta)\cos\alpha)\cos r_\theta \\ 0 \\ 1 \end{bmatrix}, \tag{4}$$

$$0 < \beta < h_p - h_{xx},$$

where

$$h_p = h - r_c\cos\alpha + y_e$$

$$y_e = \frac{r_c(\cos\alpha - 1 + \sin\alpha)(\cos\alpha + 1 - \sin\alpha)}{2\cos\alpha}$$

2-3 Regions $\overline{cd}$ and $\overline{gf}$

As shown in Figs. 3a and 3b, the two circular arc curves $\overline{cd}$ and $\overline{gf}$ of the disc cutter are used to generate the root fillet of the sun gear and the tip fillet of the planet gear. The radius of the circular-arc curve of the proposed cutter is r_c. The respective equations for regions $\overline{cd}$ and $\overline{gf}$, represented in the coordinate system S_1 can be written as:

$$\boldsymbol{R}_1^{cd} = \begin{bmatrix} x_1^{cd} \\ y_1^{cd} \\ z_1^{cd} \\ 1 \end{bmatrix} = \begin{bmatrix} (r_c\sin\theta_c + w)\cos r_\theta + (r\cos\theta_p + h\cos\alpha - r_c(1 - \cos\theta_c))\sin r_\theta \\ -(r_c\sin\theta_c + w)\sin r_\theta + (r\cos\theta_p + h\cos\alpha - r_c(1 - \cos\theta_c))\cos r_\theta \\ 0 \\ 1 \end{bmatrix}, \tag{5}$$

$$0 < \theta_c < \theta_{c1},$$

and

$$\boldsymbol{R}_1^{gf} = \begin{bmatrix} x_1^{gf} \\ y_1^{gf} \\ z_1^{gf} \\ 1 \end{bmatrix} = \begin{bmatrix} -(r_c\sin\theta_c + w)\cos r_\theta + (r\cos\theta_p + h\cos\alpha - r_c(1 - \cos\theta_c))\sin r_\theta \\ (r_c\sin\theta_c + w)\sin r_\theta + (r\cos\theta_p + h\cos\alpha - r_c(1 - \cos\theta_c))\cos r_\theta \\ 0 \\ 1 \end{bmatrix}, \tag{6}$$

$$0 < \theta_c < \theta_{c1}$$

where

$$w = r\sin\theta_p - h\sin\alpha - r_c\cos\alpha + y_e$$

$$\theta_{c1} = \frac{\pi}{2} - \alpha$$

2-4 Region $\overline{df}$

Region $\overline{df}$ is used to generate the bottom land of the sun gear or the top land of the planet gear, and ℓ_h represents a design parameter of the disc cutter. Based on Fig. 3a, the equation for the region $\overline{df}$, represented in the coordinate system S_1, can be written as:

$$\boldsymbol{R}_1^{df} = \begin{bmatrix} x_1^{df} \\ y_1^{df} \\ z_1^{df} \\ 1 \end{bmatrix} = \begin{bmatrix} \ell_h \cos r_\theta + (h\cos\alpha + r\cos\theta_p)\sin r_\theta \\ -\ell_h \sin r_\theta + (h\cos\alpha + r\cos\theta_p)\cos r_\theta \\ 0 \\ 1 \end{bmatrix}, -w < \ell_h < w \tag{7}$$

Following Eqs. (1)-(7) and using a computer program and computer-aided software, the complete contour of a disc cutter with five teeth can be obtained. As shown in Fig. 4, when the parameter w represented in Eq. (7) equals zero, a circular-arc curve on the top land of the disc cutter is obtained, where, the radius of the circular arc curve, r_c can be represented as:

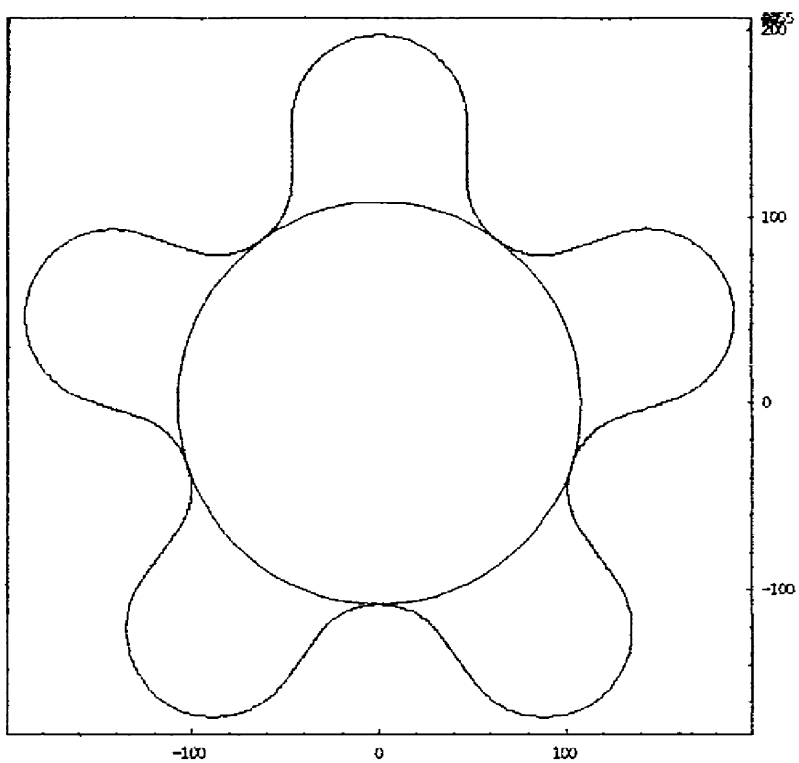

Fig. 4. The top land of the disc cutter was circular arc curve when $w = 0$.

$$r_c = \frac{2\cos\alpha(r\sin\theta_p - h\sin\alpha)}{2\cos\alpha\cos\alpha - (\cos\alpha - 1 + \sin\alpha)(\cos\alpha + 1 - \sin\alpha)} \tag{8}$$

This paper presents an imaginary disc cutter consisting of a straight-line and a circular arc curve that is potentially to generate nonstandard gears for gear pumps. This cutter is also adaptable for generating a ring gear.

3. ESTABLISHING GEOMETRIC MODEL

In this section, two applications of the proposed imaginary disc cutter are presented. One is the gear pump with a few teeth; the other is a planetary gear mechanism. First, the kinematics of the planetary gear mechanism can be described with reference to Fig. 5, where the disc cutter is a generating curve and the sun gear and the ring gear are the generated curves. Based on gear theory, the mathematical model of the two generated curves can be regarded as an envelope to the family of the generating curves.

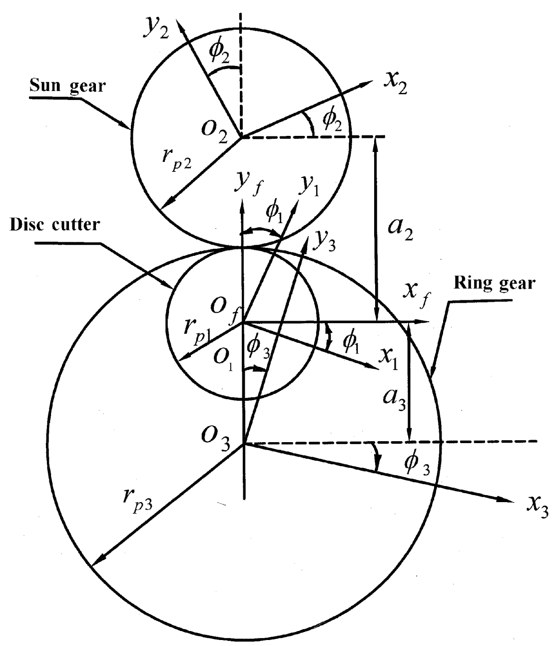

Fig. 5. The kinematics relationship among the sun gear, the ring gear and the disc cutter.

As illustrated in Fig. 5, a reference coordinate system $S_f(o_f,\ x_f,\ y_f,\ z_f)$ is attached to the gear housing while coordinate system $S_2(O_2,\ x_2,\ y_2,\ z_2)$ is attached to the sun gear. Coordinate system $S_1(O_1,\ x_1,\ y_1,\ z_1)$ is attached to the disc cutter, and coordinate system $S_3(O_3,\ x_3,\ y_3,\ z_3)$ is attached to the ring gear, where z_1, z_2, z_3 and z_f are determined by the right-hand coordinate system.

Also shown in Fig. 5, the disc cutter rotates along the z_1 axis with an angle ϕ_1, the sun gear rotates along the z_2 axis with a rotary angle ϕ_2, and the ring gear rotates along the z_3 axis with an angle ϕ_3 and the relationship between the angles ϕ_1 and ϕ_2 is given by $\phi_2 = n_1\phi_1/n_2$. The relationship between the angles ϕ_1 and ϕ_3 is given by $\phi_3 = n_1\phi_1/n_3$. The symbol n_2 is the number of teeth of the sun gear, and the symbol n_3 is the number of teeth of the ring gear.

As shown in Fig. 5, the minimum circle is the proposed disc cutter; the maximum circle is the ring gear and the middle circle is the sun gear, both of which are generated by the envelope to the family of the cutter curves. The distance between the origin o_2 and o_1 is a_2, where a_2 is equal to the sum of r_{p2} and r_{p1}. Similarly, the distance between the origin o_3 and o_1 is a_3, where a_3 is equal to the difference between r_{p3} and r_{p1}. Parameters r_{p1}, r_{p2}, and r_{p3} are the radius of the pitch circle on the disc cutter, the sun gear, and the ring gear, respectively.

As shown in Fig. 5, if the maximum circle becomes the disc cutter, the minimum circle is the envelope to the family of the cutter curves. The obtained envelopes can be used in a gear pump with a smaller number of teeth. This function of the proposed cutter will be illustrated by a numerical example. Here, the concept of an imaginary disc cutter is used to generate the gears for a pair of gear pumps.

In order to obtain the mathematical models of the sun and the ring gears, a coordinate transformation matrix from the S_1 coordinate system to the S_2 coordinate system and to the S_3 coordinate system can also be used. By applying homogeneous coordinates and a 4×4 matrix for coordinate transformation, the matrix $\boldsymbol{M}_{21}$ and $\boldsymbol{M}_{31}$, can be obtained as follows:

$$\boldsymbol{M}_{21}(\phi_1) = \begin{bmatrix} \cos(\phi_2 + \phi_1) & \sin(\phi_2 + \phi_1) & 0 & -a_2 \sin\phi_2 \\ -\sin(\phi_2 + \phi_1) & \cos(\phi_2 + \phi_1) & 0 & -a_2 \cos\phi_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{9}$$

and

$$\boldsymbol{M}_{31}(\phi_1) = \begin{bmatrix} \cos(\phi_3 - \phi_1) & -\sin(\phi_3 - \phi_1) & 0 & -a_3 \sin\phi_3 \\ \sin(\phi_3 - \phi_1) & \cos(\phi_3 - \phi_1) & 0 & a_3 \cos\phi_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{10}$$

Using equations (1)~(7) and the coordinate transformation matrix $\boldsymbol{M}_{21}(\phi_1)$ and $\boldsymbol{M}_{31}(\phi_1)$, the family of the cutter curves can be expressed by

$$\boldsymbol{R}'^{g}_{2} = \boldsymbol{M}_{21} \boldsymbol{R}^{g}_{1} \tag{11}$$

$$\boldsymbol{R}'^{g}_{3} = \boldsymbol{M}_{31} \boldsymbol{R}^{g}_{1} \tag{12}$$

where vector $\boldsymbol{R}'^{g}_{2}$ and $\boldsymbol{R}'^{g}_{3}$ are the family of the cutter curves, and superscript g is $\overline{ab}$, $\overline{bc}$ $\overline{cd}$, $\overline{df}$, $\overline{ih}$, $\overline{hg}$, and $\overline{gf}$. The position vector of the cutter is $\boldsymbol{R}^{g}_{1}$, as indicated in Section 2.

On the other hand, the equation for meshing can be obtained by first finding the relative velocity of an arbitrary contact point on the generating curve and the generated curves. Therefore, the relative velocity between the two generated curves and the normal vector on the generating curve can be expressed in a fixed coordinate system, and are written as:

$$\boldsymbol{V}^{12}_{f} = -(\dot{\phi}_2 + \dot{\phi}_1)[(x^g_1 \sin\phi_1 - y^g_1 \cos\phi_1)\boldsymbol{i}_f + (x^g_1 \cos\phi_1 + y^g_1 \sin\phi_1)\boldsymbol{j}_f] - a_2 \dot{\phi}_2 \boldsymbol{i}_f, \tag{13}$$

$$V_f^{13} = (\dot{\phi}_3 - \dot{\phi}_1)[(x_1^g \sin\phi_1 - y_1^g \cos\phi_1)\boldsymbol{i}_f + (x_1^g \cos\phi_1 + y_1^g \sin\phi_1)\boldsymbol{j}_f] - a_3\dot{\phi}_3\boldsymbol{i}_f, \tag{14}$$

and

$$N_f^1 = \boldsymbol{k}_f \times \frac{\partial \boldsymbol{R}_f^g}{\partial \beta} = (-y_{1\beta}^g \cos\phi_1 + x_{1\beta}^g \sin\phi_1)\boldsymbol{i}_f + (y_{1\beta}^g \sin\phi_1 + x_{1\beta}^g \cos\phi_1)\boldsymbol{j}_f \tag{15}$$

where V_f^{12} and V_f^{13} are the relative velocity between the generating curve and the generated curve, and represented in the fixed coordinate system S_f. N_f^1 is a normal vector to the generating curve and is represented in the fixed coordinate system S_f. Here, the generating curve is the proposed disc cutter and the generated curve is the sun gear and the ring gear. Vector $\boldsymbol{R}_f^g$ is the parameter equation $\boldsymbol{R}_1^g$ in Section 2, and is represented in the fixed coordinate system S_f. The vector $\frac{\partial \boldsymbol{R}_f^g}{\partial \beta}$ is the partial differentiation of the vector $\boldsymbol{R}_f^g$ with respect to β. Substituting Eqs. (1) and (2) into Eq. (15), the parameter β becomes the parameter θ_t in equation (15). Substituting Eqs. (5) and (6) into Eq. (15), the parameter β becomes the parameter θ_c in equation (15). Similarly, the parameter β in equation (15) becomes ℓ_h when substituted equation (7) into Eq. (15).

According to gear theory, the meshing equation can now be determined in terms of a coordinate system $\boldsymbol{S}_f$ and is expressed as:

$$N_f^1 \bullet V_f^{12} = 0 \tag{16}$$

$$N_f^1 \bullet V_f^{13} = 0 \tag{17}$$

Inserting the normal vector equation (15) and the relative velocity equations (16) and (17), the meshing equations can be written as

$$x_{1\beta}^g \sin\phi_1 - y_{1\beta}^g \cos\phi_1 + \frac{1}{a_2}(\frac{\partial\phi_1}{\partial\phi_2} + 1)(y_1^g y_{1\beta}^g + x_1^g x_{1\beta}^g) = 0 \tag{18}$$

$$x_{1\beta}^g \sin\phi_1 - y_{1\beta}^g \cos\phi_1 + \frac{1}{a_3}(\frac{\partial\phi_1}{\partial\phi_3} - 1)(y_1^g y_{1\beta}^g + x_1^g x_{1\beta}^g) = 0 \tag{19}$$

Substituting all the vector equations for regions $\overline{ab}$, $\overline{bc}$ $\overline{cd}$, $\overline{df}$, $\overline{ih}$, $\overline{hg}$, and $\overline{gf}$ in Section 2 into Eq. (18), the meshing equations for these regions for generating a sun gear can be written as:

The meshing equations for the regions $\overline{ab}$ and $\overline{ih}$ is

$$\phi_1 = \pm\theta_t - r_\theta \mp \cos^{-1}[\frac{1}{a_2}(\frac{n_2}{n_1} + 1)(r + r_{c1})\cos(\theta_p + \theta_x + \theta_t)]. \tag{20}$$

The meshing equations for the regions $\overline{bc}$ and $\overline{hg}$ is

$$\phi_1 = \pm\alpha - r_\theta \mp \cos^{-1}[\frac{1}{a_2}(\frac{n_2}{n_1} + 1)((h_p - \beta) + r\cos(\theta_p + \alpha))]. \tag{21}$$

The meshing equations for the regions $\overline{cd}$ and $\overline{gf}$ is

$$\phi_1 = \mp\theta_c - r_\theta \pm \sin^{-1}[\frac{1}{a_2}(\frac{n_2}{n_1}+1)(r\sin\theta_c\cos\theta_p + h\sin\theta_c\cos\alpha - r_c\sin\theta_c - w\cos\theta_c)]. \quad (22)$$

The meshing equations for the region $\overline{df}$ is

$$\phi_1 = -r_\theta + \sin^{-1}[-(\frac{n_2}{n_1}+1)\frac{\ell_h}{a_2}]. \quad (23)$$

Similarly, substituting Eqs. (1), (2), (3), (4), (5), (6), (7) into Eq. (19), the meshing equation for generating a ring gear can be represented as:

The meshing equation for regions $\overline{ab}$ and $\overline{ih}$ is

$$\phi_1 = \pm\theta_t - r_\theta \mp \cos^{-1}[\frac{1}{a_3}(\frac{n_3}{n_1}-1)(r+r_{c1})\cos(\theta_p+\theta_x+\theta_t)]. \quad (24)$$

The meshing equation for regions $\overline{bc}$ and $\overline{hg}$ is

$$\phi_1 = \pm\alpha - r_\theta \mp \cos^{-1}[\frac{1}{a_3}(\frac{n_3}{n_1}-1)((h_p-\beta)+r\cos(\theta_p+\alpha))]. \quad (25)$$

The meshing equation for regions $\overline{cd}$ and $\overline{gf}$ is

$$\phi_1 = \mp\theta_c - r_\theta \pm \sin^{-1}[\frac{1}{a_3}(\frac{n_3}{n_1}-1)(r\sin\theta_c\cos\theta_p + h\sin\theta_c\cos\alpha - r_c\sin\theta_c - w\cos\theta_c)]. \quad (26)$$

The meshing equation for region $\overline{df}$is

$$\phi_1 = -r_\theta + \sin^{-1}[-(\frac{n_3}{n_1}-1)\frac{\ell_h}{a_3}]. \quad (27)$$

Simultaneous consideration of equation (11) and the upper sign of meshing equation (20) indicates the tip fillet of the sun gear generated by the circular-arc curve $\overline{ab}$ of the imaginary disc cutter. Simultaneous consideration of equation (11) and the lower sign of meshing equation (20) indicates the tip fillet of the sun gear generated by the circular-arc curve $\overline{ih}$ of the imaginary disc cutter. Simultaneous consideration of equation (11) and the upper sign of meshing equation (21) indicates the working surface of the sun gear generated by straight-line segment $\overline{bc}$ of the imaginary disc cutter. By simultaneously considering (11) and the lower sign of equation meshing (21) indicates the working surface of sun gear generated by straight-line segment $\overline{hg}$ of the imaginary disc cutter. Simultaneous consideration (11) and the upper sign of meshing equation (22) indicates the root fillet of the sun gear generated by circular-arc curve $\overline{cd}$ of the imaginary disc cutter. Simultaneous consideration (11) and the lower sign of meshing equation (22) indicates the root fillet of the sun gear generated by circular-arc curve $\overline{gf}$ of the imaginary disc cutter. As shown in Fig. 5, based on Eqs. (12), (24), (25), (26) and (27), the mathematical model of the ring gear is determined. Furthermore, as indicated in Fig. 5, through the above calculations, a planet gear of the proposed planetary gear mechanism can be generated by an imaginary disc cutter when the ring gear becomes an imaginary disc cutter and the original imaginary disc cutter becomes a blank material. Hereby, Fig. 5 is transformed to Fig. 6. The

Table 1. Major design parameters of the disc cutter.

Parameters	Disc Cutter	Sun gear	Ring Gear
Number of teeth	$n_1 = 15$ (Fig. 1) $n_1 = 5$ (Fig. 6) $n_1 = 10$ (Fig. 7)	$n_2 = 3$ (Fig. 1) $n_2 = 3$ (Fig. 6) $n_2 = 20$ (Fig. 7)	$n_3 = 3$ (Fig. 1) $n_3 = 3$ (Fig. 6) $n_3 = 40$ (Fig. 7)
Modified Coefficient η	1.05	1	1
Pitch radius	$r_{p1} = r + \frac{0.5h}{\cos\alpha}$	$r_{p2} = \frac{n_2}{n_1} r_{p1}$	$r_{p3} = \frac{n_3}{n_1} r_{p1}$
r	108mm	108mm	108mm
pressure angle α	20° (Fig. 1) 0° (Fig. 6) 5° (Fig. 7)	20° (Fig. 1) 0° (Fig. 6) 5° (Fig. 7)	20° (Fig. 1) 0° (Fig. 6) 5° (Fig. 7)
Height of tooth h	$30mm$ (Fig. 1) $110mm$ (Fig. 6) $40mm$ (Fig. 7)	$30mm$ (Fig. 1) $110mm$ (Fig. 6) $40mm$ (Fig. 7)	$30mm$ (Fig. 1) $110mm$ (Fig. 6) $40mm$ (Fig. 7)
r_c	$10mm$(Fig. 1) Eq. (8) [Figs. 6 & 7]	$10mm$(Fig. 1) Eq. (8) [Figs. 6 & 7]	$10mm$(Fig. 1) Eq. (8) [Figs. 6 & 7]
Distance of center		$a_2 = r_{p1} + r_{p2}$	$a_3 = r_{p3} - r_{p1}$
θ_x	$\pi/(60\eta)$ (Fig. 1)	$\pi/(60\eta)$ (Fig. 1)	$\pi/(60\eta)$ (Fig. 1)
$\theta_p = \left(\frac{\pi}{n_1} - \theta_x\right) \Big/ \eta$	$\pi/(18\eta)$ (Fig. 6) $\pi/(36\eta)$ (Fig. 7)	$\pi/(18\eta)$ (Fig. 6) $\pi/(36\eta)$ (Fig. 7)	$\pi/(18\eta)$ (Fig. 6) $\pi/(36\eta)$ (Fig. 7)

detailed mathematical model of the planet gear is similar to the above process, but for simplification, the imaginary disc cutter in Fig. 5 is adopted to the planet gear.

The dimensional parameters of the proposed disc cutter are listed in Table 1. To demonstrate the contours of the proposed planetary gear mechanism, the *Turbo C++* programming language and the *SolidWorks* software package are utilized to draw the complete profile of the gear mechanism, and the solid models are then transferred into ANSYS Workbench. To obtain a gear pump, the top land of the disc cutter with circular arc curve is used. In other words, the condition of equation (8) is used in the design of the disc cutter. As shown in Fig. 6, the imaginary disc cutter with five numbers of teeth is used to illustrate the generating process of the gear pump. A gear pump with a few teeth is then obtained. The generated gears with three numbers of teeth is obtained and shown in Fig. 6. The sun gear is generated by the cutter in Fig. 2a while the planet gear is generated by the cutter in Fig. 2b.

4. THE UNDERCUTTING CONDITION OF THE IMAGINARY DISC CUTTER

A singular point may be assumed to appear on the generated gear created by the imaginary disc cutter. A parametric representation of the proposed cutter is presented in section 2, and equations of meshing are presented in Section 3. However, whether the generated curve is a regular curve remains a variable. Moreover, it is noted that equations (16) and (17) are also satisfied by singular points of the family of the cutter curves. Therefore, the singular points should be excluded from the family of the cutter curves and the generated gears.

Here, the region $\overline{bc}$ is used to illustrate how to determine the singular point on generated gear. Conditions of non-undercutting by an imaginary disc-cutter can be determined by using the general approach that was represented by Litvin [18].

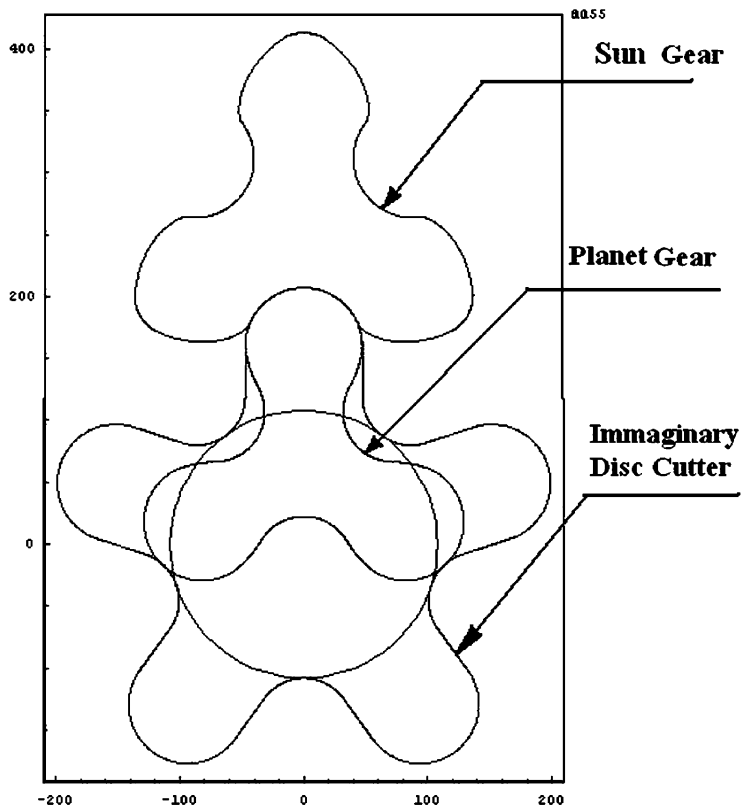

Fig. 6. The gear pump with small numbers of teeth generated by the imaginary disc cutter ($\theta_x = \dfrac{10\pi}{180\eta}$).

Let parameter r_θ be equal to zero in Eq. (3). The proposed imaginary disc-cutter can be displayed a two-dimensional system. Then

$$\boldsymbol{R}_1^{bc} = \begin{bmatrix} x_1^{bc} \\ y_1^{bc} \end{bmatrix} = \begin{bmatrix} r\sin\theta_p - (h_p - \beta)\sin\alpha \\ r\cos\theta_p + (h_p - \beta)\cos\alpha \end{bmatrix} \tag{28}$$

$$\boldsymbol{R}_{1\beta}^{bc} = \frac{\partial \boldsymbol{R}_1^{bc}}{\partial \beta} = \begin{bmatrix} x_{1\beta}^{bc} \\ y_{1\beta}^{bc} \end{bmatrix} = \begin{bmatrix} \sin\alpha \\ -\cos\alpha \end{bmatrix} \tag{29}$$

$$\boldsymbol{N}_1^1 = \boldsymbol{k} \times \boldsymbol{R}_{1\beta}^{bc} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \tag{30}$$

$$\boldsymbol{V}_1^{12} = -(\dot{\phi}_1 + \dot{\phi}_2)\begin{bmatrix} -(r\cos\theta_p + (h_p - \beta)\cos\alpha \\ r\sin\theta_p - (h_p - \beta)\sin\alpha \end{bmatrix} - \dot{\phi}_2 \begin{bmatrix} a_2\cos\phi_1 \\ a_2\sin\phi_1 \end{bmatrix} \tag{31}$$

Meshing equation for generating sun gear can be written as

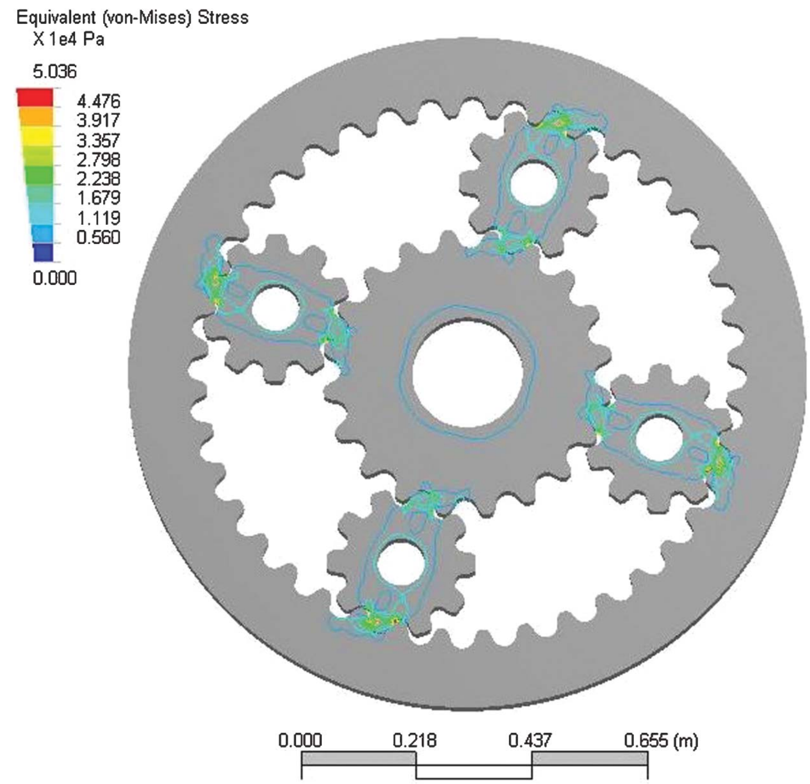

Fig. 7. The distribution of von-Mises stress for the proposed planetary gear mechanism (thickness=50mm, $n_1=10$, $\eta=1$, $n_2=20$, $n_3=40$, $\alpha=\pi/36$, h=40, r=108, $\theta_x=\frac{5\pi}{180\eta}$).

$$f(\beta,\quad \phi_1)=\boldsymbol{N}_1^1 \bullet \boldsymbol{V}_1^{12}=\cos(\alpha-\phi_1)-\frac{1}{a_2}(1+\frac{\partial\phi_1}{\partial\phi_2})[r\cos(\theta_p+\alpha)+(h_p-\beta)]=0 \tag{32}$$

where $\boldsymbol{V}_1^{12}$ is the relative velocity between the generating curve and the generated curve, and represented in the coordinate system S_1. $\boldsymbol{N}_1^1$ is a normal vector to the generating curve and is represented in the coordinate system S_1. Eq. (32) and the upper sign of meshing equation (21) are the same formula for meshing equation. Based on Litvin's gear theory [18], a requirement for determining the singularity of generated gear can be represented by

$$\Delta_1=\begin{vmatrix} \frac{\partial x_1^{bc}}{\partial\beta} & -V_{1x}^{12} \\ f_\beta & -f_{\phi_1}\frac{d\phi_1}{dt} \end{vmatrix}=0 \tag{33}$$

$$\Delta_2=\begin{vmatrix} \frac{\partial y_1^{bc}}{\partial\beta} & -V_{1y}^{12} \\ f_\beta & -f_{\phi_1}\frac{d\phi_1}{dt} \end{vmatrix}=0 \tag{34}$$

$$\Delta_3 = \begin{vmatrix} \frac{\partial x_1^{bc}}{\partial \beta} & -V_{1x}^{12} \\ \frac{\partial y_1^{bc}}{\partial \beta} & -V_{1y}^{12} \end{vmatrix} = 0 \tag{35}$$

Eq. (35) yields the equation of meshing $f(\beta, \quad \phi_1) = 0$ and is not applied for investigation of singularities. Eq. (33) or Eq. (34) can be applied to obtain determination of singularities. Substituting Eqs. (28) and (31) into Eq. (33), the following function can be obtained.

$$g(\beta, \quad \phi_1) = \sin\alpha \sin(\alpha - \phi_1)\frac{\partial \phi_1}{\partial \phi_2} + \frac{1}{a_2}(1 + \frac{\partial \phi_1}{\partial \phi_2})\{(1 + \frac{\partial \phi_1}{\partial \phi_2})[r\cos\theta_p + (h_p - \beta)\cos\alpha]\} = 0 \tag{36}$$

Rearranging Eq.(36), the parameter β can be written as

$$\beta = h_p - \frac{-\sin\alpha\sin(\alpha - \phi_1)\frac{\partial \phi_1}{\partial \phi_2} + \cos\phi_1(1 + \frac{\partial \phi_1}{\partial \phi_2}) - r\cos\theta_p \frac{1}{a_2}(1 + \frac{\partial \phi_1}{\partial \phi_2})^2}{\frac{1}{a_2}(1 + \frac{\partial \phi_1}{\partial \phi_2})^2 \cos\alpha} \tag{37}$$

By simultaneous consideration of Eqs. (32) and (37), the values of parameters β and ϕ_1 can be determined and substituted into vector $\boldsymbol{R}'^{bc}_2$ of Eq. (11). Then, the singular points on the generated gear are obtained.

5. STRESS AND DEFORMATION ANALYSIS

In the above sections, the geometrical models of the planetary gear mechanism were developed using the *Turbo C++* programming language and a computer aided design (CAD) computer program, and then transferred to the FE package *ANSYS Workbench* for contact stress analysis. Based on finite methodology [17] and the application of a general proposed computer program [16], a stress analysis is performed. A computer program running on a *Windows XP* operating system is used to evaluate the numerical solution for the contact stress. The material is assumed to be isotropic and homogeneous. In this case, the material used is steel with a Young's modulus $E = 207GPa$ and a Poisson ratio of $\upsilon = 0.3$.

Fig. 7 was first obtained by a planar profile under Turbo C++ programming language and SolidWorks software package secondly this profile was extruded a distance. The thickness of planetary gear mechanism is given 50mm. This study assumed that there have no assembly errors. A FEA software package, ANSYS Workbench was applied to evaluate the stress distribution of the proposed planetary gear mechanism. Four additional pins within the planet gears were referenced to its own center. A torque 5Nm was applied at the planet gears rotational axis to create surface contact between the sun gear and the ring gear tooth. The ring gear was fixed at the sun cylindrical surface. In the contact stress analysis, the operator had to set the "contact pair" as the "contact bodies" and the "target bodies". Here, the contact bodies are identified as the planet gear which the target bodies were identified as the sun and the ring gear. The option, which was no seperation, was specified to define the interaction between the contact surfaces. The stress distribution of the proposed planetary gear was evaluated and the results shown in Fig. 7. Finite elements were used to mesh the solid geometry of the planetary gear mechanism, and the distribution of the von-Mises stress is shown in Fig. 7.

6. CONCLUSIONS

The geometric model of a novel disc cutter is presented in this study. The mathematical models of generated gears are derived. Gear theory and differential geometry are used to obtain the parametric expression of the disc cutter and the meshing equation. Based on the developed mathematical models, a three-dimensional stress analysis of the proposed gear is investigated. A circular-arc curve on the top land of the disc cutter is determined. In this work, the proposed gear with circular-arc tooth is nonstandard gear. Through the method of an imaginary disc cutter, a gear pair with a few teeth can be established.

According to the developed mathematical model of the proposed gear, the non-undercutting condition of an imaginary disc cutter is also determined. Furthermore, through the proposed mathematical model, a computer program which is applicable to the assembly of geometric models of gear transmission ratio with parameters limit of teeth is developed.

In closing, in this study, the proposed mathematical model for a disc cutter should be very useful in the design and production of a gear pump and a transmission mechanism.

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NOTATION

a_2	distance between o_1 and o_2
a_3	distance between o_1 and o_3
h	tooth length of a tooth
η	modified coefficient or backlash
r	base radius of a disc cutter
r_{p1}	radius of pitch circle of a disc cutter
r_{p2} and r_{p3}	radius of pitch circle of generated gears
n_1	number of teeth of a disc cutter
n_2 and n_3	number of teeth of a generated gear
$\boldsymbol{N}_f^1$	normal vector to the generating curve
$\mathbf{M}_{21}$ and $\mathbf{M}_{31}$	co-ordinate transformation matrix from co-ordinate system $_1S_1$ to co-ordinate systems S_2 and S_3
$\boldsymbol{R}_1^g$	position vector $\boldsymbol{R}_1^g$ where the upper sign indicates regions $\overline{ab}$, $\overline{bc}$ $\overline{cd}$, $\overline{df}$, $\overline{ih}$, $\overline{hg}$, and $\overline{gf}$ of the proposed cutter
$\boldsymbol{R}'^g_2$ and $\boldsymbol{R}'^g_3$	the family of the generating curves.
S_i(o_i , x_i, y_i, z_i)	co-ordinate systems where subscript i=1, 2, 3, f. 1 denotes disc cutter, 2 denotes sun gear, 3 denote ring gear, f is rigidly connected to the frame of reference.
$\boldsymbol{V}_f^{12}$ and $\boldsymbol{V}_{\mathrm{f}}^{13}$	relative velocity between the generating curve and two generated curves represented in coordinate system S_f
α	pressure angle.
$\theta_p+\theta_x$	half angle of one tooth of the proposed disc cutter.
ϕ_1 ϕ_2, and ϕ_3	respectively allowed to rotate angles about z_1-axis, z_2-axis, and z_3-axis.
θ_t, θ_c, β, and ℓ_h	curvilinear parameters for regions $\overline{ab}$, $\overline{bc}$, $\overline{cd}$ and $\overline{df}$, respectively.