# KINEMATIC ISOTROPY OF THE H4 CLASS OF PARALLEL MANIPULATORS

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## Abstract

This paper presents the isotropic conditions for the topological class of H4 parallel manipulators with an articulated traveling plate which has four degrees of freedom. First, a generic kinematic model of this class of manipulators is developed, then we impose isotropic conditions on the Jacobian matrix. From the newly obtained equations, design constraints and a design procedure allowing the determination of all isotropic geometries are obtained. The proposed procedure allows the successive choice and computation of each and all geometrical parameters of an isotropic manipulator of the H4 class.

**Keywords:** isotropy; parallel manipulator; traveling plate; kinematics.

# ISOTROPIE DES MANIPULATEURS PARALLÈLES DE LA CLASSE H4

## Résumé

Cet article présente les conditions d'isotropie de la classe topologique des manipulateurs parallèles à nacelle articulée H4 possédant quatre degrés de liberté. Un modèle cinématique générique de cette classe de manipulateurs est d'abord développé, puis on impose une condition isotrope à la matrice jacobienne. Des équations obtenues on trouve les contraintes et une procédure de conception permettant de déterminer toutes les géométries isotropes. La procédure de design proposée permet de choisir et de calculer successivement chacun des paramètres géométriques d'un manipulateur isotrope de classe H4.

**Mots-clés :** isotropie; manipulateur parallèle; nacelle articulée; cinématique.

## 1. INTRODUCTION

We wish to determine the isotropic conditions of parallel manipulators belonging to the H4 topological class of manipulators.

As defined in [1]: a topological class is the group of mechanisms having the same topology regardless of the geometry. The topology describes the arrangement of the joints of the manipulator while the geometry describes the relative localization of the joints on the links. A necessary and sufficient number of geometric parameters is required to describe in a unique way all the geometries of a topological class.

The modelling and analysis of geometrical conditions making a given class of manipulator isotropic have been proposed for manipulators of the Star class [2] and Delta class [3], for example.

The H4 parallel manipulator with an articulated travelling plate (see Fig. 2) has been developed at the LIRMM (Laboratoire d'informatique, de robotique et de microélectronique de Montpellier). Its topology is illustrated in Fig. 1, where $R$ is a revolute joint and $S$ is a spherical joint. This manipulator has four degrees of freedom: three for translation and one for rotation [4]. Parallel manipulators like the H4 are specially interesting because of their complementary characteristics to serial manipulators.

The isotropic conditions we look for allow the manipulator to move its end-effector at equal speeds in all directions from commands of equal intensity. This property is particularly interesting because the manipulator display, at this state, its best kinematics performances. Moreover, the isotropic postures are far from the singularities.

In order to identify the isotropic conditions, we must search the conditions to be applied to the Jacobian matrices. Since the Jacobian matrices depend both on the posture and the geometry, it is difficult to have a geometry which is isotropic at all postures. Therefore, only some particular geometries can reach isotropy, and only for one or a few postures.

We are thus searching for the conditions that will allow us to obtain isotropic geometries, not only isotropic postures of a specific geometry.

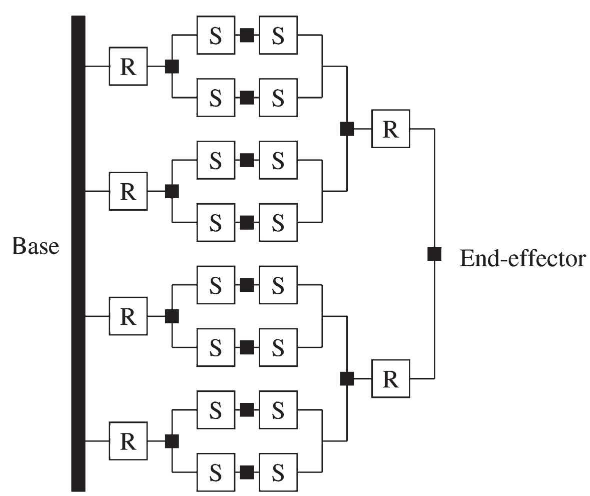


Fig. 1. Topology of the H4 manipulator.

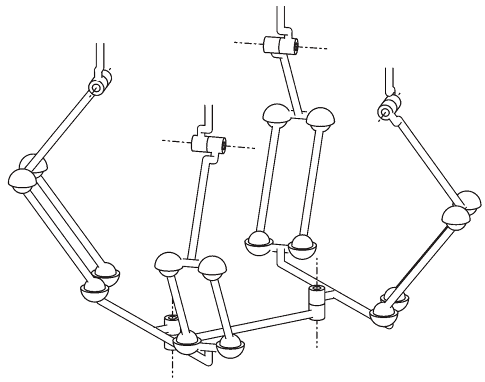

Fig. 2. Architecture of the H4 manipulator.

## 2. MODELLING OF THE H4 CLASS

Referring to Fig. 3, the points $\{A_i\}_1^4$ are fixed and attached to the base while the point $P$ is attached to the end-effector and is therefore mobile. The joints at points $A_i$ and $D_i$ are revolute while the joints at points $B_i$ and $C_i$ are spherical. The links between points $A_i$ and $B_i$ rotate about the axis $\hat{\boldsymbol{u}}_i$ by an angle $q_i$. The end-effector at $P$ rotates by an angle $\theta$ about an axis normal to the plane generated by the relative movement of points $C_i$. Without loss of generality, this axis is chosen as being the unitary vector $\hat{\boldsymbol{k}}$ in the global frame.

The links between the spherical joints at points $B_i$ and $C_i$, represented by vector $\boldsymbol{r}_i$, form a $\Pi$ joint which propagates the orientation of axis $\hat{\boldsymbol{u}}_i$ from point $A_i$ to point $C_i$. The bars linking the point $C_1$ to the point $C_4$ and $C_2$ to $C_3$ always keep the same orientation. When they move one about another, the bars always remain in the same plane so that the rotation of the end-effector located at point $P$ is always about the axis of $\hat{\boldsymbol{k}}$ which always maintains the same orientation.

The position of point $A_i$ belonging to the base and the position of the end-effector $P$ are both known relatively to a global frame. Knowing the angle $q_i$ of the motorized revolute joint located at $A_i$, it is possible to find the position of $B_i$. From the position and angle $\theta$ of the end-effector $P$, we can find the position of $C_i$ (see Fig. 4).

Since points $C_i$ and $B_i$ are connected by a rigid link, these two points are mathematically related by a rigidity condition. Indeed, the norm of vector $\boldsymbol{r}_i$ going from point $B_i$ to point $C_i$ is constant and equal to $r_i$.

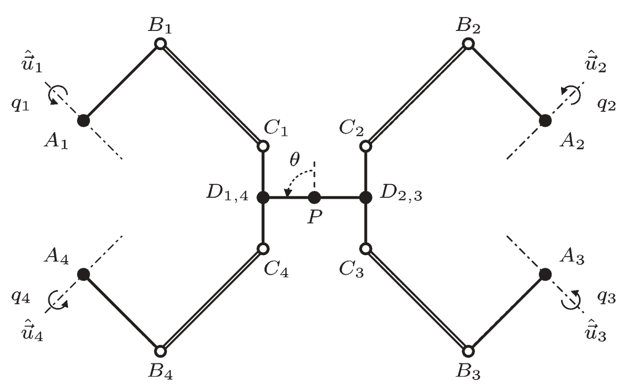


Fig. 3. Schematic diagram of the H4 manipulator.

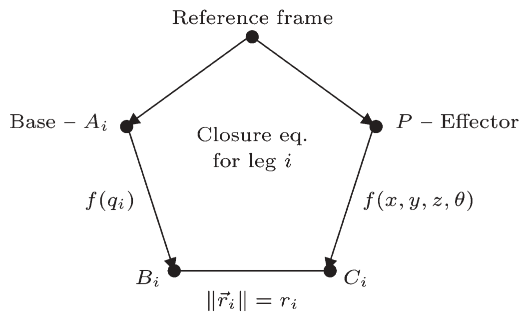


Fig. 4. Schematic diagram of the closure equation for each leg.

### 2.1. Rigidity Condition

The distance $r_i$ between points $B_i$ and $C_i$ belongs to the same rigid link, and hence, is constant, i.e.,

$$\boldsymbol{r}_i^{\top}\boldsymbol{r}_i = r_i^2 \tag{1}$$

Deriving Eq. (1) with respect to time, the rigidity condition expressed in terms of speed is given as:

$$\dot{\boldsymbol{b}}_i^{\top}\boldsymbol{r}_i = \dot{\boldsymbol{c}}_i^{\top}\boldsymbol{r}_i \tag{2}$$

where $\dot{\boldsymbol{b}}_i$ and $\dot{\boldsymbol{c}}_i$ are respectively the speed of points $B_i$ and $C_i$ expressed in the same reference frame. This expression represents the equiprojectivity of the speeds $\dot{\boldsymbol{b}}_i$ and $\dot{\boldsymbol{c}}_i$ on the axis $\boldsymbol{r}_i$ linking points $B_i$ and $C_i$ belonging the same rigid link.

### 2.2. Closure Equations

From Fig. 5, it is possible to write the closure equation of the kinematic loop associated to leg $i$:

$$\boldsymbol{b}_i = \boldsymbol{a}_i + \boldsymbol{p}_i(q_i) \tag{3}$$

$$\boldsymbol{c}_i = \boldsymbol{p} - \boldsymbol{t}_i(\theta) - \boldsymbol{s}_i \tag{4}$$

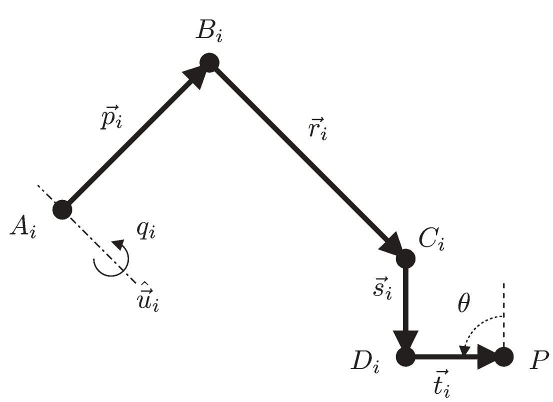


Fig. 5. An isolated leg.

$$\boldsymbol{r}_i = \boldsymbol{c}_i - \boldsymbol{b}_i \tag{5}$$

where $\boldsymbol{a}_i$, $\boldsymbol{b}_i$, $\boldsymbol{c}_i$ and $\boldsymbol{p}$ are respectively the position vectors of points $A_i$, $B_i$, $C_i$ and $P$. The vector $\boldsymbol{p}_i$ is a function of $q_i$ and the vector $\boldsymbol{t}_i$ is a function of $\theta$. Substituting Eq. (3) and Eq. (4) in Eq. (5), we obtain:

$$\boldsymbol{r}_i = \boldsymbol{p} - \boldsymbol{t}_i(\theta) - \boldsymbol{s}_i - \boldsymbol{a}_i - \boldsymbol{p}_i(q_i) \tag{6}$$

Then substituting Eq. (6) in Eq. (1), we have:

$$r_i^2 = \boldsymbol{r}_i^\top (\boldsymbol{p} - \boldsymbol{t}_i(\theta) - \boldsymbol{s}_i - \boldsymbol{a}_i - \boldsymbol{p}_i(q_i)) \tag{7}$$

Deriving Eq. (7) with respect to time:

$$\boldsymbol{r}_i^\top \left( \dot{\boldsymbol{p}} + \left( \boldsymbol{t}_i \times \hat{\boldsymbol{k}} \right) \dot{\theta} - (\boldsymbol{p}_i \times \hat{\boldsymbol{u}}_i) \dot{q}_i \right) = 0 \tag{8}$$

where, for the sake of brevity, the positive directions of $\dot{\theta}$ and $\dot{q}_i$ are chosen so that we don't have to carry minus signs in Eq. (10), and where $\hat{\boldsymbol{k}}$ is recalled to be a unitary vector parallel to the axis of rotation of the platform. Using the distributivity property of the dot product, Eq. (8) can be rewritten separating the terms $\dot{\boldsymbol{p}}$, $\dot{\theta}$ and $\dot{q}_i$:

$$\boldsymbol{r}_i^\top \dot{\boldsymbol{p}} + \boldsymbol{r}_i^\top \left( \boldsymbol{t}_i \times \hat{\boldsymbol{k}} \right) \dot{\theta} = \boldsymbol{r}_i^\top (\boldsymbol{p}_i \times \hat{\boldsymbol{u}}_i) \dot{q}_i \tag{9}$$

Equation (9) links the speed of the motorized joints $\dot{q}_i$ located at the base of the legs to the speeds of the end-effector $\dot{\boldsymbol{p}}$ and $\dot{\theta}$. Writing down Eq. (9) for all the four legs, separating $\dot{\boldsymbol{p}}$ and $\dot{\theta}$, we have the following system of equations:

$$\boldsymbol{A}\dot{\boldsymbol{x}} = B\dot{q}, \qquad \dot{\boldsymbol{x}} \equiv \begin{bmatrix} \dot{\boldsymbol{p}} & \dot{\theta} \end{bmatrix}^\top, \qquad \dot{\boldsymbol{q}} \equiv [\dot{q}_1 \quad \dot{q}_2 \quad \dot{q}_3 \quad \dot{q}_4]^\top \tag{10}$$

where

$$\boldsymbol{A} \equiv \begin{bmatrix} \boldsymbol{r}_1^T & \boldsymbol{r}_1^\top \left( \boldsymbol{t}_1 \times \hat{\boldsymbol{k}} \right) \\ \boldsymbol{r}_2^T & \boldsymbol{r}_2^\top \left( \boldsymbol{t}_2 \times \hat{\boldsymbol{k}} \right) \\ \boldsymbol{r}_3^T & \boldsymbol{r}_3^\top \left( \boldsymbol{t}_3 \times \hat{\boldsymbol{k}} \right) \\ \boldsymbol{r}_4^T & \boldsymbol{r}_4^\top \left( \boldsymbol{t}_4 \times \hat{\boldsymbol{k}} \right) \end{bmatrix}, \qquad \boldsymbol{B} \equiv \begin{bmatrix} \boldsymbol{r}_1^\top (\boldsymbol{p}_1 \times \hat{\boldsymbol{u}}_1) & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \boldsymbol{r}_4^\top (\boldsymbol{p}_4 \times \hat{\boldsymbol{u}}_4) \end{bmatrix} \tag{11}$$

and $\dot{\boldsymbol{x}}$ is the velocity of the end-effector, $\dot{\boldsymbol{q}}$ the velocity vector of the motorized joints, $\boldsymbol{A}$ is of dimension $4 \times 4$ and $\boldsymbol{B}$ is diagonal and also $4 \times 4$.

## 3. PROBLEM FORMULATION

We wish to determine the isotropic conditions on the Jacobian matrices of Eq. (10), but before this we need to render them adimensional.

### 3.1. Adimensionalisation

A manipulator that can both position and orient itself in space has dimensionally non homogeneous Jacobian matrices because they involve dimensional lengths and adimensional angles. The nonhomogeneity of the Jacobian matrices is eliminated by introducing a characteristic length [5, 6]. In order to obtain dimensionally homogeneous matrices, we need to divide both sides of Eq. (10) by a characteristic length in order to render them adimensional.

Using $L$ as the unit of length and $T$ as the unit of time, both sides of Eq. (10) have $L^2/T$ as unit. A further analysis reveals that the Jacobian matrix $\boldsymbol{A}$ has components with dimensions $L$ and $L^2$, while the matrix $\boldsymbol{B}$ has $L^2$ components.

In an adimensional form, when $\lambda$ is taken as the natural length, the Jacobian matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ can be rewritten as:

$$\boldsymbol{A} \equiv \begin{bmatrix} \boldsymbol{r}_1^T/\lambda & \boldsymbol{r}_1^\top\left(\boldsymbol{t}_1 \times \hat{\boldsymbol{k}}\right)/\lambda^2 \\ \boldsymbol{r}_2^T/\lambda & \boldsymbol{r}_2^\top\left(\boldsymbol{t}_2 \times \hat{\boldsymbol{k}}\right)/\lambda^2 \\ \boldsymbol{r}_3^T/\lambda & \boldsymbol{r}_3^\top\left(\boldsymbol{t}_3 \times \hat{\boldsymbol{k}}\right)/\lambda^2 \\ \boldsymbol{r}_4^T/\lambda & \boldsymbol{r}_4^\top\left(\boldsymbol{t}_4 \times \hat{\boldsymbol{k}}\right)/\lambda^2 \end{bmatrix} \quad \boldsymbol{B} \equiv \begin{bmatrix} \dfrac{\boldsymbol{r}_1^\top(\boldsymbol{p}_1 \times \hat{\boldsymbol{u}}_1)}{\lambda^2} & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & \dfrac{\boldsymbol{r}_4^\top(\boldsymbol{p}_4 \times \hat{\boldsymbol{u}}_4)}{\lambda^2} \end{bmatrix} \tag{12}$$

### 3.2. Isotropic Condition

The isotropic condition on the Jacobian matrices is expressed as:

$$\dot{\boldsymbol{x}}^\top \dot{\boldsymbol{x}} = 1 \tag{13}$$

It constrains the velocity of the end-effector in all directions to a velocity of unit magnitude, i.e. a unit sphere of dimension $m$, where $m$ is the total number of degrees of freedom of the manipulator. A Jacobian matrix is said isotropic when it transforms a unit sphere from the $m$-dimensional space of the end-effector to a $n$-dimensional sphere in the joint space scaled by a factor $\alpha$.

Substituting Eq. (10) in Eq. (13), we obtain:

$$\dot{\boldsymbol{q}}^\top \left(\boldsymbol{A}^{-1}\boldsymbol{B}\right)^\top \boldsymbol{A}^{-1}\boldsymbol{B}\,\dot{\boldsymbol{q}} = 1 \tag{14}$$

which represents a velocity ellipsoid in the adimensional joint space. The matrix $\boldsymbol{A}^{-1}$ $\boldsymbol{B}$ is therefore isotropic if its singular values are all identical and different from zero:

$$\boldsymbol{C}^\top \boldsymbol{C} = \alpha^2 1 \tag{15}$$

where $\boldsymbol{C}^\top \equiv \boldsymbol{B}^{-1}\boldsymbol{A}$

### 3.3. Isotropy of the Matrix *C*

Matrix *C* can be expressed as:

$$C^\top \equiv \begin{bmatrix} \lambda \boldsymbol{r}_1^\top / g_1 & h_1/g_1 \\ \lambda \boldsymbol{r}_2^\top / g_2 & h_2/g_2 \\ \lambda \boldsymbol{r}_3^\top / g_3 & h_3/g_3 \\ \lambda \boldsymbol{r}_4^\top / g_4 & h_4/g_4 \end{bmatrix} \tag{16}$$

where $g_i \equiv \boldsymbol{r}_i^\top (\boldsymbol{p}_i \times \hat{\boldsymbol{u}}_i)$ and $h_i \equiv \boldsymbol{r}_i^\top \left(\boldsymbol{t}_i \times \hat{\boldsymbol{k}}\right)$.

The isotropic conditions of matrix *C* given in Eq. (15) are orthogonality conditions. The rows of an orthogonal matrix form an orthonormal basis. All rows have the same norm and are mutually orthogonal. It is automatically the same for the columns which also form an orthonormal basis.

#### 3.3.1. Orthogonality Conditions

The dot product between two rows (or columns) of *C* must vanish, i.e.,

$$\left[\lambda \boldsymbol{r}_i^\top / g_i \quad h_i/g_i\right] \left[\lambda \boldsymbol{r}_j^\top \big/ g_j \quad h_j/g_j\right]^\top = 0 \tag{17}$$

When developed:

$$\left(\frac{\lambda \boldsymbol{r}_i}{g_i}\right) \cdot \left(\frac{\lambda \boldsymbol{r}_j}{g_j}\right) + \left(\frac{h_i}{g_i}\right)\left(\frac{h_j}{g_j}\right) = 0 \tag{18}$$

Simplifying $g_i$ and developing $h_i$ results in:

$$\lambda^2 \boldsymbol{r}_i \cdot \boldsymbol{r}_j + \left(\boldsymbol{r}_i^\top \left(\boldsymbol{t}_i \times \hat{\boldsymbol{k}}\right)\right)\left(\boldsymbol{r}_j^\top \left(\boldsymbol{t}_j \times \hat{\boldsymbol{k}}\right)\right) = 0 \tag{19}$$

then dividing by $\|\boldsymbol{r}_i\| \, \|\boldsymbol{r}_j\|$ we obtain:

$$\lambda^2 \hat{\boldsymbol{r}}_i \cdot \hat{\boldsymbol{r}}_j + \left(\hat{\boldsymbol{r}}_i^\top \left(\boldsymbol{t}_i \times \hat{\boldsymbol{k}}\right)\right)\left(\hat{\boldsymbol{r}}_j^\top \left(\boldsymbol{t}_j \times \hat{\boldsymbol{k}}\right)\right) = 0 \tag{20}$$

Vectors $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ of Eq. (20) become unitary as $\hat{\boldsymbol{r}}_i$ and $\hat{\boldsymbol{r}}_j$. Using the definitions of the dot product and the cross product, while fixing (without loss of generality) $\sin \angle_{\hat{\boldsymbol{k}}}^{\boldsymbol{t}_i} = 1$ which appears in the cross products:

$$-\lambda^2 \cos \angle_{\hat{\boldsymbol{r}}_i}^{\hat{\boldsymbol{r}}_j} = t_i \cos \angle_{\hat{\boldsymbol{r}}_i}^{\boldsymbol{t}_i \times \hat{\boldsymbol{k}}} t_j \cos \angle_{\hat{\boldsymbol{r}}_j}^{\boldsymbol{t}_j \times \hat{\boldsymbol{k}}} \tag{21}$$

We can rewrite Eq. (21) for all pairs of rows of *C* as

$$\begin{aligned} -\sigma_{1,2} = \frac{t_{1,4}}{\lambda}\mu_1 \frac{t_{2,3}}{\lambda}\mu_2 \quad -\sigma_{1,3} = \frac{t_{1,4}}{\lambda}\mu_1 \frac{t_{2,3}}{\lambda}\mu_3 \quad -\sigma_{1,4} = \frac{t_{1,4}}{\lambda}\mu_1 \frac{t_{1,4}}{\lambda}\mu_4 \\ -\sigma_{2,3} = \frac{t_{2,3}}{\lambda}\mu_2 \frac{t_{2,3}}{\lambda}\mu_3 \quad -\sigma_{2,4} = \frac{t_{2,3}}{\lambda}\mu_2 \frac{t_{1,4}}{\lambda}\mu_4 \quad -\sigma_{3,4} = \frac{t_{2,3}}{\lambda}\mu_3 \frac{t_{1,4}}{\lambda}\mu_4 \end{aligned} \tag{22}$$

where $\mu_i \equiv \cos \angle_{\hat{\boldsymbol{r}}_i}^{\boldsymbol{t}_i \times \hat{\boldsymbol{k}}} \equiv \hat{\boldsymbol{r}}_i^\top \left(\hat{\boldsymbol{t}}_i \times \hat{\boldsymbol{k}}\right)$ and $\sigma_{i,j} \equiv \angle_{\hat{\boldsymbol{r}}_i}^{\hat{\boldsymbol{r}}_j} \equiv \hat{\boldsymbol{r}}_i^\top \hat{\boldsymbol{r}}_j$. From the orthogonality conditions, we have six equations from the non diagonal elements of the identity matrix. The $t_i$ appear divided by λ, they are therefore normalized. Because $\boldsymbol{t}_1$ and $\boldsymbol{t}_4$ represent the same vector, they are written as $\boldsymbol{t}_{1,4}$. The same applies for $\boldsymbol{t}_2$ and $\boldsymbol{t}_3$ which are written as $\boldsymbol{t}_{2,3}$.

Defining $\beta_i \equiv \frac{t_i}{\lambda}\mu_i$, Eq. (22) becomes:

$$\begin{array}{lll} -\sigma_{1,2} = \beta_1\beta_2 & -\sigma_{1,3} = \beta_1\beta_3 & -\sigma_{1,4} = \beta_1\beta_4 \\ -\sigma_{2,3} = \beta_2\beta_3 & -\sigma_{2,4} = \beta_2\beta_4 & -\sigma_{3,4} = \beta_3\beta_4 \end{array} \tag{23}$$

From which we find the following constraint:

$$\sigma_{1,2}\sigma_{3,4} = \sigma_{1,3}\sigma_{2,4} = \sigma_{1,4}\sigma_{2,3} \tag{24}$$

### 3.3.2. Normality conditions

All the rows (or columns) of $\boldsymbol{C}$ must have the same norm:

$$\left[\lambda \boldsymbol{r}_i^\top / g_i \quad h_i/g_i\right]\left[\lambda \boldsymbol{r}_i^\top / g_i \quad h_i/g_i\right]^\top = \alpha^2 \tag{25}$$

When developed:

$$\left(\frac{\lambda \boldsymbol{r}_i}{g_i}\right)\cdot\left(\frac{\lambda \boldsymbol{r}_i}{g_i}\right) + \left(\frac{h_i}{g_i}\right)\left(\frac{h_i}{g_i}\right) = \alpha^2 \tag{26}$$

By moving the $g_i$ on the other side of the equality, we obtain:

$$\lambda^2 \boldsymbol{r}_i \cdot \boldsymbol{r}_i + \left(\boldsymbol{r}_i^\top\left(\boldsymbol{t}_i \times \hat{\boldsymbol{k}}\right)\right)^2 = \alpha^2\left(\boldsymbol{r}_i^\top\left(\boldsymbol{p}_i \times \hat{\boldsymbol{u}}_i\right)\right)^2 \tag{27}$$

then dividing by $\|\boldsymbol{r}_i\|^2$ results in:

$$\lambda^2 + \left(\hat{\boldsymbol{r}}_i^\top\left(\boldsymbol{t}_i \times \hat{\boldsymbol{k}}\right)\right)^2 = \alpha^2\left(\hat{\boldsymbol{r}}_i^\top\left(\boldsymbol{p}_i \times \hat{\boldsymbol{u}}_i\right)\right)^2 \tag{28}$$

Vectors $\boldsymbol{r}_i$ of Eq. (28) become unitary. Using the definition of the dot product and the cross product, while fixing (without loss of generality) $\sin \angle_{\boldsymbol{k}_i}^{\boldsymbol{t}_i} = 1$ and $\sin \angle_{\boldsymbol{p}_i}^{\boldsymbol{u}_i} = 1$ which appear in the cross products:

$$\lambda^2 + t_i^2 \cos^2 \angle_{\hat{\boldsymbol{r}}_i}^{\boldsymbol{t}_i \times \hat{\boldsymbol{k}}} = \alpha^2 p_i^2 \cos^2 \angle_{\hat{\boldsymbol{r}}_i}^{\boldsymbol{p}_i \times \hat{\boldsymbol{u}}_i} \tag{29}$$

We can rewrite Eq. (29) for all rows of $\boldsymbol{C}$ as:

$$\begin{array}{ll} 1 + \frac{{t_{1,4}}^2}{\lambda^2}{\mu_1}^2 = \alpha^2\frac{{p_1}^2}{\lambda^2}{\eta_1}^2 & 1 + \frac{{t_{2,3}}^2}{\lambda^2}{\mu_2}^2 = \alpha^2\frac{{p_2}^2}{\lambda^2}{\eta_2}^2 \\ 1 + \frac{{t_{2,3}}^2}{\lambda^2}{\mu_3}^2 = \alpha^2\frac{{p_3}^2}{\lambda^2}{\eta_3}^2 & 1 + \frac{{t_{1,4}}^2}{\lambda^2}{\mu_4}^2 = \alpha^2\frac{{p_4}^2}{\lambda^2}{\eta_4}^2 \end{array} \tag{30}$$

where $\mu_i \equiv \cos \angle_{\hat{\boldsymbol{r}}_i}^{\boldsymbol{t}_i \times \hat{\boldsymbol{k}}} \equiv \hat{\boldsymbol{r}}_i^\top \left(\hat{\boldsymbol{t}}_i \times \hat{\boldsymbol{k}}\right)$ and $\eta_i \equiv \angle_{\hat{\boldsymbol{r}}_i}^{\boldsymbol{p}_i \times \hat{\boldsymbol{u}}_i} \equiv \hat{\boldsymbol{r}}_i^\top (\hat{\boldsymbol{p}}_i \times \hat{\boldsymbol{u}}_i)$. From the normality conditions we have four equations from the diagonal elements of the identity matrix. Moreover $t_i$ and $p_i$ appear divided by λ, they are therefore normalized.

#### 3.3.3. Relation Between the $\sigma_{i,j}$

When the orthogonality condition for the $\hat{\boldsymbol{r}}_i$ is written in a scalar form, the angles between the $\hat{\boldsymbol{r}}_i$ appear in the form of $\sigma_{i,j}$. The $\hat{\boldsymbol{r}}_i$ can be seen as a radius on a unit sphere, since there are four of them we thus have six angles subtended by the arcs between those radii. A strong relation exists between these angles which must be considered. Using the arcs to make triangles on the sphere we can thus use spherical trigonometry. It is sufficient to know five of these angles to be able to determine the absolute value of the 6th angle using the law of spherical cosines [7].

The relationship that the $\sigma_{i,j}$ must conform to is:

$$\arccos\left(\frac{\sigma_{1,2} - \sigma_{1,4}\sigma_{2,4}}{\kappa_{1,4}\kappa_{2,4}}\right) \pm \arccos\left(\frac{\sigma_{1,3} - \sigma_{1,4}\sigma_{3,4}}{\kappa_{1,4}\kappa_{3,4}}\right) \pm \arccos\left(\frac{\sigma_{2,3} - \sigma_{2,4}\sigma_{3,4}}{\kappa_{2,4}\kappa_{3,4}}\right) = 0 \tag{31}$$

where $\sigma_{i,j} \equiv \cos \angle_{\hat{\boldsymbol{r}}_i}^{\hat{\boldsymbol{r}}_j}$ and $\kappa_{i,j} \equiv \sin \angle_{\hat{\boldsymbol{r}}_i}^{\hat{\boldsymbol{r}}_j}$.

Moreover, Eq. (31) has real solutions when the following constraints are satisfied:

$$-1 \leq \frac{\sigma_{1,2} - \sigma_{1,4}\sigma_{2,4}}{\kappa_{1,4}\kappa_{2,4}} \leq 1 \qquad -1 \leq \frac{\sigma_{1,3} - \sigma_{1,4}\sigma_{3,4}}{\kappa_{1,4}\kappa_{3,4}} \leq 1 \qquad -1 \leq \frac{\sigma_{2,3} - \sigma_{2,4}\sigma_{3,4}}{\kappa_{2,4}\kappa_{3,4}} \leq 1 \tag{32}$$

## 4. ISOTROPIC SOLUTIONS

In summary, there are 11 equations, see Eqs. (22), (30), and (31), associated with the isotropic condition applied to matrix $\boldsymbol{C}$. It is possible to reduce this system of equations to only one equation by substituting Eqs. (22) and (30) in Eq. (31). However, the choice of the parameters appearing in these equations is made easier when considering the system as a whole rather than as a single equation.

### 4.1. Design procedure

It is possible to obtain an isotropic manipulator by choosing and calculating successively each and all of the parameters appearing in the isotropy equations as we propose in the following procedure:

**Choice of** $\sigma_{1,4}$: We first choose the value of $\sigma_{1,4}$. Since $\sigma_{i,j}$ are cosines, $\sigma_{1,4}$ must be chosen within the interval [-1, 1].

**Choice of** $\sigma_{2,4}$: From Eq. (23), we know that:

$$\beta_1 = -\frac{\sigma_{1,4}}{\beta_4} \quad \text{and} \quad \beta_2 = -\frac{\sigma_{2,4}}{\beta_4} \tag{33}$$

$$\text{so} \quad -\sigma_{1,2} = \frac{\sigma_{1,4}\sigma_{2,4}}{{\beta_4}^2} \quad \text{and} \quad \frac{\sigma_{1,2}}{|\sigma_{1,2}|} = -\left(\frac{\sigma_{1,4}}{|\sigma_{1,4}|}\right)\left(\frac{\sigma_{2,4}}{|\sigma_{2,4}|}\right) \tag{34}$$

The sign of $\sigma_{1,2}$ must be the inverse of the sign obtained from the product of $\sigma_{1,4}$ and $\sigma_{2,4}$. Applying the sign constraint of $\sigma_{1,2}$ to the constraints of Eq. (32), we see that:

$$|\kappa_{1,4}\kappa_{2,4}| > |\sigma_{1,4}\sigma_{2,4}| \tag{35}$$

From the definition of $\sigma_{i,j}$ and $\kappa_{i,j}$ at Eq. (31), and knowing that $\kappa_{i,j} = \sqrt{1-\sigma_{i,j}{}^2}$ from trigonometric identities, we can thus isolate $\sigma_{2,4}$ and find that $\sigma_{2,4}$ must be chosen within $\left[1-\sqrt{1-\sigma_{1,4}{}^2}, +\sqrt{1-\sigma_{1,4}{}^2}\right]$.

**Choice of** $\sigma_{1,2}$: From the conditions (32), we must choose $\sigma_{1,2}$ in the interval $\left[0,\ \sigma_{1,4}\sigma_{2,4} + \sqrt{1-\sigma_{1,4}{}^2}\sqrt{1-\sigma_{2,4}{}^2}\right]$ or $\left[0,\ \sigma_{2,4} - \sqrt{1-\sigma_{1,4}{}^2}\sqrt{1-\sigma_{2,4}{}^2}\right]$ depending upon the sign of $\sigma_{1,2}$, see Eq. (34).

**Computation of** $\sigma_{1,3}$: Again from Eq. (24), we can express $\sigma_{2,3}$ and $\sigma_{3,4}$ as a function of $\sigma_{1,2}$, $\sigma_{1,3}$, $\sigma_{1,4}$ and $\sigma_{2,4}$ as:

$$\sigma_{2,3} = \frac{\sigma_{1,3}\sigma_{2,4}}{\sigma_{1,4}}; \qquad \sigma_{3,4} = \frac{\sigma_{1,3}\sigma_{2,4}}{\sigma_{1,2}} \tag{36}$$

Substituting Eq. (36) in Eq. (31), we obtain an equation solely function of $\sigma_{1,2}$, $\sigma_{1,3}$, $\sigma_{1,4}$ and $\sigma_{2,4}$. Thus $\sigma_{1,3}$ is solved for with a computer algebra system:

$$\begin{aligned}
\sigma_{1,3} = \pm\big(&\sigma_{1,4}{}^2\big(-\sigma_{2,4}{}^4+\sigma_{2,4}{}^2-\sigma_{1,4}{}^4-2\sigma_{1,4}{}^2\sigma_{2,4}{}^2-\sigma_{1,4}{}^2\sigma_{1,2}{}^2-2\sigma_{1,2}\sigma_{1,4}\sigma_{2,4}-\sigma_{2,4}{}^2\sigma_{1,2}{}^2\\
&+2\sigma_{1,2}{}^3\sigma_{1,4}\sigma_{2,4}-4\sigma_{1,2}{}^2\sigma_{1,4}{}^2\sigma_{2,4}{}^2+4\sigma_{1,2}\sigma_{1,4}\sigma_{2,4}{}^3+4\sigma_{1,2}\sigma_{1,4}{}^3\sigma_{2,4}+\sigma_{1,4}{}^2\big)\big)^{(1/2)}\\
&\sigma_{1,2}\Big(\big(-\sigma_{1,4}{}^2+2\sigma_{1,2}\sigma_{1,4}\sigma_{2,4}-\sigma_{2,4}{}^2\big)(\sigma_{1,2}-\sigma_{1,4}\sigma_{2,4})^2\Big/\big(2\sigma_{1,2}{}^3\sigma_{1,4}\sigma_{2,4}-3\sigma_{1,2}{}^2\sigma_{1,4}{}^2\sigma_{2,4}{}^2\\
&-\sigma_{1,4}{}^2\sigma_{1,2}{}^2-\sigma_{2,4}{}^2\sigma_{1,2}{}^2+2\sigma_{1,2}\sigma_{1,4}{}^3\sigma_{2,4}+2\sigma_{1,2}\sigma_{1,4}\sigma_{2,4}{}^3-\sigma_{1,4}{}^2\sigma_{2,4}{}^2\big)\Big)^{(1/2)}\\
&\Big/\big(2\sigma_{1,2}{}^2\sigma_{1,4}\sigma_{2,4}-2\sigma_{1,2}\sigma_{1,4}{}^2\sigma_{2,4}{}^2-\sigma_{1,2}\sigma_{2,4}{}^2-\sigma_{1,2}\sigma_{1,4}{}^2+\sigma_{1,4}{}^3\sigma_{2,4}+\sigma_{1,4}\sigma_{2,4}{}^3\big)
\end{aligned} \tag{37}$$

The sign of $\sigma_{1,3}$ can be freely chosen.

**Computation of** $\sigma_{2,3}$ **and** $\sigma_{3,4}$: Knowing all other $\sigma_{i,j}$, we can easily calculate $\sigma_{2,3}$ and $\sigma_{3,4}$ with Eq. (36).

**Computation of** $\beta_i$: From the definition $\beta_i$ and Eq. (23), we can write $\beta_1$ as a function of the $\sigma_{i,j}$:

$$\beta_1 = \pm\sqrt{\frac{-\sigma_{1,2}\sigma_{1,3}}{\sigma_{2,3}}} \tag{38}$$

The sign of $\beta_1$ can be chosen freely. We can then calculate the other $\beta_i$ easily as:

$$\beta_2 = -\frac{\sigma_{1,2}}{\beta_1}, \qquad \beta_3 = \frac{\sigma_{1,3}}{\beta_1}, \qquad \beta_4 = -\frac{\sigma_{1,4}}{\beta_1} \tag{39}$$

**Computation of** $\mu_i$ **and** $\hat{\boldsymbol{r}}_i$: Since $\beta_1$ and $\beta_4$ share the same $t_{1,4}$ and that the same applies for $\beta_2$ and $\beta_3$ which share the same $t_{2,3}$, then we can combine them into the following two relationships:

$$\frac{\beta_1}{\mu_1} = \frac{\beta_4}{\mu_4} \qquad \frac{\beta_2}{\mu_2} = \frac{\beta_3}{\mu_3} \tag{40}$$

So, if we combine Eq. (40) to the definitions of $\beta_i$ (see Eq. (23)) and $\mu_i$ (see Eq. (22)):

$$\hat{\boldsymbol{r}}_i^\top \hat{\boldsymbol{r}}_j = -\beta_i\beta_j \qquad \mu_i = \hat{\boldsymbol{r}}_i^\top\left(\hat{\boldsymbol{t}}_i \times \hat{\boldsymbol{k}}\right) \tag{41}$$

We then have a system of equations in terms of $\hat{\boldsymbol{r}}_i$, $\beta_i$, $\mu_i$ and $\hat{\boldsymbol{t}}_i \times \hat{\boldsymbol{k}}$.

We are looking for a solution to this system. Without loss of generality, we choose $\hat{\boldsymbol{t}}_i \times \hat{\boldsymbol{k}}$ as being along the $y$ axis. In order to have only one possible solution, we choose to restrain $\hat{\boldsymbol{r}}_1$ to be in the positive quadrant of the $y$ - $z$ plane and to restrain $\hat{\boldsymbol{r}}_2$ to have a positive component along the $x$-axis. We can then numerically find the $\hat{\boldsymbol{r}}_i$ and the $\mu_i$ that satisfy these conditions. All the rotations of the vectors $\hat{\boldsymbol{r}}_i$ about the $y$-axis and their reflexions about the $x$, $y$ and $z$ axis are also solutions.

**Computation of** $t_i/\lambda$: Having found $\mu_i$, we can determine the values of $t_{1,4}/\lambda$ and $t_{2,3}/\lambda$ from the definition of the $\beta_i$, see Eq. (23):

$$t_{1,4}/\lambda = \beta_1/\mu_1 \qquad t_{2,3}/\lambda = \beta_2/\mu_2 \tag{42}$$

**Choice of** $\alpha$: The amplification factor $\alpha$ can be freely chosen as a positive non-zero value.

**Choice of** $\eta_i$: The $\eta_i$ are cosines, we can choose them freely in the interval [-1, 1].

**Computation of** $p_i/\lambda$: From Eq. (30), we can the find $p_i/\lambda$ as:

$$\begin{aligned} \frac{p_1}{\lambda} &= \sqrt{\frac{1+(t_{1,4}/\lambda)^2\mu_1^2}{\alpha^2\eta_1^2}} \quad & \frac{p_2}{\lambda} &= \sqrt{\frac{1+(t_{2,3}/\lambda)^2\mu_2^2}{\alpha^2\eta_2^2}} \\ \frac{p_3}{\lambda} &= \sqrt{\frac{1+(t_{2,3}/\lambda)^2\mu_3^2}{\alpha^2\eta_3^2}} \quad & \frac{p_4}{\lambda} &= \sqrt{\frac{1+(t_{1,4}/\lambda)^2\mu_4^2}{\alpha^2\eta_4^2}} \end{aligned} \tag{43}$$

**Other parameters:** All the parameters in the equations associated to the isotropic condition have been determined. The other parameters in the kinematics equations can then be determined. These parameters are the components of the vectors $\boldsymbol{s}_i$, the norm of the vectors $\boldsymbol{r}_i$ and the orientation of the vectors $\boldsymbol{p}_i$ about the vectors $\boldsymbol{r}_i$. They can be freely chosen because they have no impact on the local isotropy. Obviously, other criteria can be used to choose them such as workspace or joint motion range, for example.

## 5. NUMERICAL EXAMPLE

Using our design procedure we can now generate isotropic manipulators belonging to the H4 class. Choosing arbitrary values along the design process, knowing that the design procedure is guaranteed to always give an isotropic solution, we obtain the following isotropic geometry shown in Fig. 6:

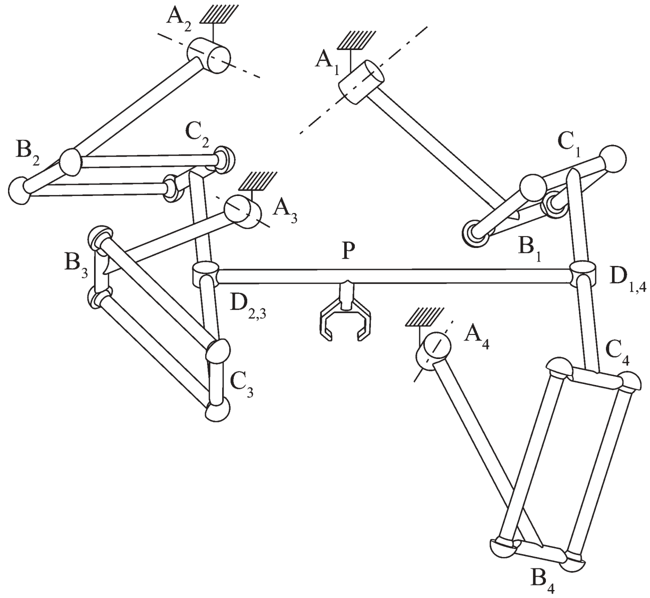


Fig. 6. An arbitrary isotropic parallel manipulator of the H4 class.

The numerical values of the parameters of this isotropic parallel manipulator are as follow:

| | | |
|---|---|---|
| $\sigma_{1,2}$ = -0.2843 | $\sigma_{1,3}$ = 0.4969 | $\sigma_{1,4}$ = 0.4390 |
| $\sigma_{2,3}$ = 0.2095 | $\sigma_{2,4}$ = 0.1850 | $\sigma_{3,4}$ = -0.3234 |
| $\mu_1$ = -0.6361 | $\eta_1$ = 0.6747 | $p_1$ = 1.2211 |
| $\mu_2$ = -0.4469 | $\eta_2$ = 0.5720 | $p_2$ = 1.1779 |
| $\mu_3$ = 0.7811 | $\eta_3$ = 0.8733 | $p_3$ = 0.8521 |
| $\mu_4$ = 0.4140 | $\eta_4$ = 0.5722 | $p_4$ = 1.2617 |
| $r_1$ = 1.0000 | $r_3$ = 1.0000 | $t_{1,4}$ = 1.2910 |
| $r_2$ = 1.0000 | $r_4$ = 1.0000 | $t_{2,3}$ = 0.7746 |
| $\alpha$ = 1.5706 | $\lambda$ = 1.0000 | |

The vectors describing the positions of points and the orientation of axes are shown as:

| | |
|---|---|
| $A_1$ = [-0.1691, 1.7192, -0.1202] | $A_2$ = [0.7137, 1.2196, -0.6353] |
| $A_3$ = [0.6083, -0.2428, -0.5920] | $A_4$ = [-0.4456, -0.5505, 0.1249] |
| $B_1$ = [-1.0415, 1.6361, 0.7301] | $B_2$ = [1.6211, 0.5531, -0.2892] |
| $B_3$ = [1.3655, -0.2189, -0.2019] | $B_4$ = [-0.9967, -1.4140, 0.8614] |
| $C_1$ = [-1.2910, 1.0000, 0.0000] | $C_2$ = [0.7746, 1.0000, 0.0000] |
| $C_3$ = [0.7746, -1.0000, 0.0000] | $C_4$ = [-1.2910, -1.0000, 0.0000] |
| $D_{1,4}$ = [-1.2910, 0.0000, 0.0000] | $D_{2,3}$ = [0.7746, 0.0000, 0.0000] |
| $\hat{u}_1$ = [0.6990, -0.0254, 0.7147] | $\hat{u}_2$ = [-0.5033, -0.8226, -0.2647] |
| $\hat{u}_3$ = [0.3585, -0.6650, -0.6551] | $\hat{u}_4$ = [-0.1514, -0.5837, -0.7977] |
| $P$ = [ 0.0000, 0.0000, 0.0000] | $\theta$ = 0.0000 |
| $\hat{k}$ = [0.0000, 0.0000, 1.0000] | |

## 6. CONCLUSION

There are infinitely many isotropic geometries for the parallel manipulators of the topological class H4. The isotropic conditions have been formulated and a design procedure for the selection and the computation of the parameters of the manipulator has been proposed. This procedure allows choosing and computing successively all geometrical parameters of any isotropic manipulators of the H4 class.

## REFERENCES


1. Jee-Hwan Ryu, editor. *Parallel Manipulators: New Developments*, chapter 4. I-Tech Education and Publishing, Vienna, Austria, 2008.
2. Baron, L. and Bernier, G., *The design of parallel manipulators of star topology under isotropic constraint.* ASME DETC, Pittsburgh, USA, 2001.
3. Baron, L., Wang, X. and Cloutier, G., *The Isotropic Conditions of Parallel Manipulators of Delta Topology.* ARK, Caldes de Malavalla, Spain, 2002.
4. Company, O., Marquet, F. and Pierrot, F., "A new high-speed 4-dof parallel robot synthesis and modeling issues," *IEEE Transactions on Robotics and Automation*, Vol. 19, No. 3, pp. 411–420, 2003.
5. Ma, O. and Angeles, J., "Optimum architecture design of platform manipulators." *Advanced Robotics, 1991. 'Robots in Unstructured Environments', 91 ICAR., Fifth International Conference on*, Vol. 2, pp. 1130–1135, Jun 1991.
6. Jorge Angeles. "The design of isotropic manipulator architectures in the presence of redundancies," *The International Journal of Robotics Research*, Vol. 11, No. 3, pp. 196–201, 1992.
7. Max Kurtz. *Handbook of Applied Mathematics for Engineers and Scientists*. McGraw-Hill, New York, 1991.