DEXTEROUS WORKSPACE OF A 3-PRRR KINEMATICALLY REDUNDANT PLANAR PARALLEL MANIPULATOR

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ABSTRACT

In this work, the dexterous workspace of a general geometry 3-PRRR kinematically redundant planar parallel manipulator with six actuated joints, three of which are redundant, is determined. The 3-PRRR manipulator is an adaptation of the 3-RRR manipulator with a redundant prismatic actuator added to each leg. Obtaining the dexterous workspace by discretizing a large area around the manipulator and determining if each point is in the workspace is relatively simple though computationally inefficient. This work proposes a geometrical method to determine the dexterous workspace of a 3-PRRR planar parallel manipulator. With this method, an exact solution of the workspace boundaries is obtained. The geometrical method uses the four-bar mechanism analogy to determine the dexterous workspace. Though the method is applied to a 3-PRRR planar manipulator, it can be readily applied to any n-PRRR planar manipulator, where *n* is the number of chains.

Keywords: dexterous workspace; kinematic redundancy; planar parallel manipulator.

ESPACE DEXTRE D'UN MANIPULATEUR 3-PRRR PARALLÈLE PLAN AVEC REDONDANCE CINÉMATIQUE

RÉSUMÉ

Ce travail consiste à déterminer l'espace dextre d'un manipulateur parallèle plan avec redondance cinématique (3-PRRR). Ce manipulateur est adapté du manipulateur 3-RRR avec l'ajout d'une articulation prismatique actionnée à chaque patte. Obtenir l'espace dextre en discrétisant une surface englobant le manipulateur est relativement simple, mais inefficace du point de vue du temps de calcul. Ce travail propose une méthode géométrique pour obtenir l'espace dextre du manipulateur 3-PRRR. Avec cette méthode, une solution exacte des frontières de l'espace dextre est obtenue. Cette méthode se base sur l'analogie des mécanismes à quatre barres pour déterminer l'espace dextre. La méthode présentée peut être facilement appliquée à n'importe quel manipulateur n-PRRR plan, où *n* indique le nombre de pattes.

Mots-clés : espace de travail dextre; redondance cinématique; manipulateur parallèle plan.

1. INTRODUCTION

The shape and size of the dexterous workspace, sometimes referred to as the primary workspace, is a useful criterion to compare dimensions of a given architecture or even to compare different architectures. The dexterous workspace can be defined as the area where a manipulator is able to reach with any orientation. In other words, for planar manipulators, any position where the end-effector (tool) can rotate 2π rad is part of the dexterous workspace.

Since the dexterous workspace is an important design criterion, much research has been conducted in this field. The 3-RRR planar parallel manipulator has been studied by several researchers. Gosselin and Angeles [1] proposed several design criteria for a symmetric 3-RRR, one of which being the dexterous workspace which was defined by two concentric circles for each leg. The dexterous workspace of the manipulator consists of the area of the intersection of the concentric circles of the three legs. Williams and Reinholtz [2] developed a geometrical method to determine and optimise the dexterous workspace once again defined by two concentric circles for each leg. Kumar [3] used screw theory to determine the controllably dexterous workspace which is the dexterous workspace devoid of any discontinuities. Pennock and Kassner [4] also treated the controllably dexterous workspace for a general geometry manipulator instead of its symmetrical counterpart. Zhaohui and Zhonghe [5] introduced a third concentric circle using four-bar linkage analogy defining a second dexterous workspace near the base of each leg. The manipulator was a symmetric 3-RRR manipulator.

Ebrahimi et al. [6] introduced kinematic redundancy to the 3-RRR manipulator to counterbalance certain drawbacks of parallel manipulators. Since there are an infinity of solutions to the inverse kinematic problem of kinematically redundant manipulators, singular configurations can readily be avoided. Kinematic redundancy can also produce a larger workspace. The dexterous workspace was obtained by discretizing an area around the manipulator and determining if each point was in the workspace. However, this method is computationally inefficient and is only an approximation where the precision is dictated by the number of points used. In this work, a geometrical method of establishing the dexterous workspace of a general geometry 3-PRRR planar parallel manipulator is developed. For the sake of brevity, the term workspace will always denote the dexterous workspace in what follows.

2. FOUR-BAR MECHANISM

The studied architecture is a parallel planar manipulator with a total of six actuated joints, three of which are redundant, and is shown in Fig. 1. The length of each link is defined as L_{ij} which is the length of the i^{th} link of the j^{th} leg and D_i represents the stroke of the prismatic actuator of leg i. Since the workspace of a parallel manipulator is the intersection of the workspace of each leg, it is simpler to establish the workspace of each leg individually and determine their intersection afterwards.

The determination of the workspace of each leg can be simplified even further by first considering an RRR manipulator, then adding the effect of the prismatic actuator. A leg of RRR configuration is shown in Fig. 2. For a given position of the end-effector (X_2, Y_2), when one considers this position to be fixed, the leg is equivalent to the familiar four-bar mechanism. If the point (X_2, Y_2) is in the workspace, the link L_3 (platform) must be able to rotate 2π radians. The workspace of each leg is the area encompassing all the points where this is possible. In what follows, links L_1, L_2 and L_3 will be considered as fixed length links (FL links) and L_0 will be considered as a variable length link (VL link).

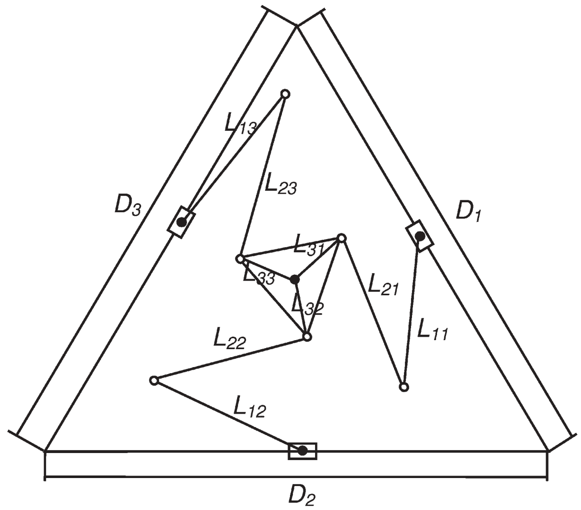


Fig. 1. Kinematically redundant 3-PRRR planar parallel manipulator [6].

The four-bar mechanism analogy is valid for any orientation of L_0. As the end-effector changes position, the base length (L_0) of the four-bar mechanism changes and is thus a variable. Each value of L_0 where L_3 can rotate 2π rad thus generates a workspace circle. Establishing the area of the workspace is then reduced to a matter of determining the limits on the values of L_0 that permit a full revolution of L_3.

The well-known Grashof's criterion is given for reference in Table 1 [7] where S is the length of the shortest link and L is that of the longest link. M_1 and M_2 are the lengths of the two remaining links. Figure 3 illustrates each four-bar mechanism category.

Of these categories, only three permit a complete revolution of L_3 and are of interest when determining the workspace of a leg, i.e., double crank, crank-rocker and change point, where L_3 must be a crank. Since the lengths of the three FL links are known, the first step in determining the workspace of a leg is to establish the shortest of the three FL links. From this, several cases arise producing different classes of workspaces (shown in Fig. 4). The different classes shown are defined and explained in the following subsections.

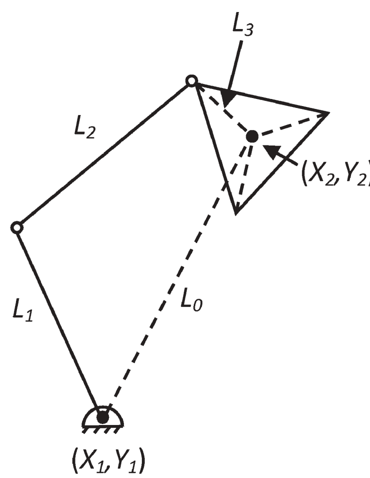


Fig. 2. One leg without the prismatic actuator (RRR).

Table 1. Categories of four-bar mechanisms.

Category	Criterion	Shortest link	Name of category	Figure
1	$L + S < M_1 + M_2$	L_0	Double crank	3(a)
2	$L + S < M_1 + M_2$	L_1 or L_3	Crank-rocker	3(b)
3	$L + S < M_1 + M_2$	L_2	Double rocker	3(c)
4	$L + S = M_1 + M_2$	Any	Change point	3(d)
5	$L + S > M_1 + M_2$	Any	Triple rocker	3(e)

2.1. Case where L_3 is the shortest FL link

Only legs with L_3 as the shortest FL link can respect the conditions for all three categories of interest with L_3 as a crank. The criterion for a crank-rocker mechanism (category 2) is:

$$L+S<M_1+M_2 \tag{1}$$

When L_0 is the longest link, Eq. (1) becomes:

$$L_0<L_1+L_2-L_3 \tag{2}$$

If Eq. (2) is not met, the mechanism falls into a triple rocker category and no links are able to complete a revolution. When the left-hand and right-hand sides of Eq. (2) are equal, this corresponds to a change point category and the outer limit of the workspace. This defines a radius r_3 henceforth corresponding to an outer limit of crank-rocker mechanisms. When the end-effector is moved towards the actuated revolute joint (X_1, Y_1), it is equivalent to the link L_0

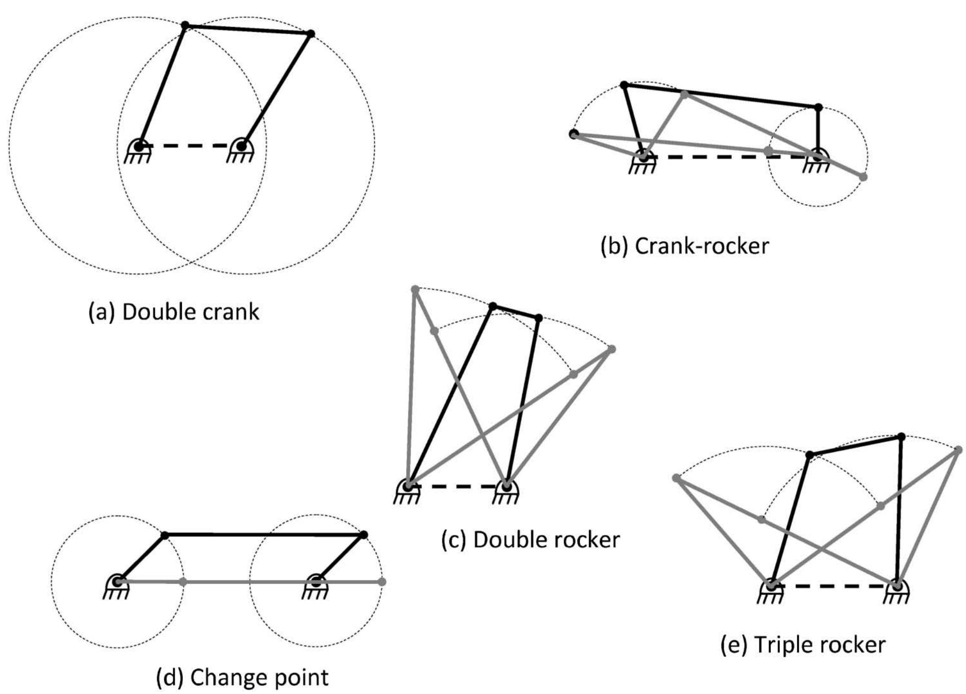


Fig. 3. Categories of four-bar mechanisms (adapted from [7]).

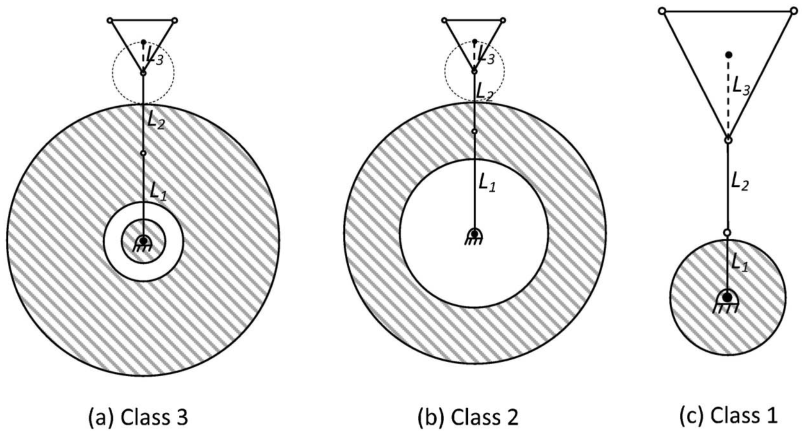


Fig. 4. Classes of the workspace of each leg without the influence of the prismatic actuator.

becoming shorter until it becomes shorter than either M_1 or M_2 (in this case, L_1 or L_2). Equation (1) then becomes:

$$L_0 > L_3 + |L_1 - L_2| \quad (3)$$

The absolute value in Eq. (3) makes it unimportant whether L_1 or L_2 is the longest. When the left-hand and right-hand sides of Eq. (3) are equal, a radius r_2 is defined which corresponds to an inner limit of crank-rocker mechanisms. Any value of L_0 bounded by radii r_2 and r_3 is within the workspace.

A double crank mechanism is also possible. When L_0 is shortest, there can be another workspace near the base of the leg. When L_0 is the shortest link, Eq. (1) becomes:

$$L_0 < L_3 - |L_1 - L_2| \quad (4)$$

When both sides of Eq. (4) are equal, another radius, r_1, is obtained which corresponds to an outer limit of double crank mechanisms. A leg with a workspace defined by three concentric circles is henceforth defined as Class 3 and is shown in Fig. 4(a).

Not every four-bar mechanism has a second workspace inside the one described above. Since every link must have a positive valued length, a zero length for L_0 in Eq. (4) represents a condition that must be satisfied for the inner workspace (r_1) to exist:

$$0 < L_3 - |L_1 - L_2| \quad (5)$$

Furthermore, if this condition is not met, one of the FL links is longer than the sum of the other two and the end-effector is unable to get closer than $L_3 - |L_1 - L_2|$ from its base (X_1, Y_1) with any orientation. A leg of this type has only two concentric circles defining its workspace and is deemed a Class 2 as shown in Fig. 4(b).

2.2. Case where L_3 is not the shortest FL link

The only link able to complete a revolution in a mechanism of the crank-rocker category is the shortest, therefore, when L_3 is not the shortest FL link, the only category of interest is that

Table 2. Dimensions and radii of legs in Fig. 4.

Figure	L_1	L_2	L_3	r_1	r_2	r_3
4(a)	1.37	1.23	0.48	0.34	0.62	2.12
4(b)	1.60	0.90	0.46	N/A	1.16	2.04
4(c)	1.00	1.40	1.30	0.90	N/A	N/A

of double crank. In this case, VL link L_0 must be the shortest link and Eq. (1) becomes:

$$L_0 < M_1 + M_2 - L \tag{6}$$

A leg of this configuration has a maximum of one circle defining its workspace and is thus considered a Class 1, defined by a radius r_1 (outer limit of double crank mechanisms). Figure 4(c) shows an example of its representation.

The end effector of each leg shown in Fig. 4 is located outside the workspace and is shown as such to illustrate an example of how each class is determined based on the lengths of the FL links. The dimensions of the example legs shown in this figure as well as the resulting radii defining the workspace are summarized in Table 2.

It is possible for a mechanism to be unable to satisfy Eq. (6), in which case the leg and thus the entire manipulator has no workspace at all. This occurs if L_3 is not the shortest FL link *and* if the longest FL link is longer than the sum of the other two.

If the two shortest links have equal lengths, Eq. (1) cannot be met unless VL link L_0 is shorter than the three FL links. Equation (6) thus also applies to the case when the two shortest FL links have equal lengths.

2.3. Transition from one four-bar mechanism category to another

As the end-effector moves and length L_0 varies, the mechanism falls into different four-bar mechanism categories. Figure 5 shows the transition from one category to another for a Class 3 leg. This figure also illustrates the radius of each concentric circle defining the Class 3 workspace, r_1, r_2 and r_3. A Class 1 workspace is defined by r_1 alone, a Class 2 by r_2 and r_3 and a Class 3 by r_1, r_2 and r_3.

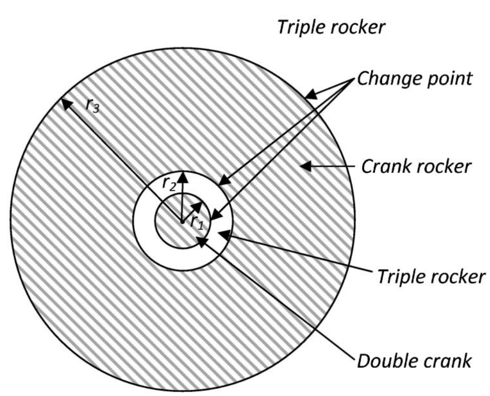


Fig. 5. Transition from one category of four-bar mechanisms to another.

Table 3. Conditions for each workspace class.

	No FL link longer than sum of others	One FL link longer than sum of others
L_3 shortest FL link	r_1, r_2 and r_3 (Class 3)	r_2 and r_3 (Class 2)
L_3 not shortest FL link	r_1 (Class 1)	No dexterous workspace

Table 3 summarizes the conditions for which the different classes presented will exist, indicating which circle radius contributes to the workspace. The lengths indicated in the table refer to the FL links (L_1, L_2 and L_3).

3. EFFECT OF THE PRISMATIC ACTUATOR ON THE WORKSPACE

The effect of adding a redundant prismatic actuator is relatively simple in nature though it introduces different types of workspaces. The limited range of motion (stroke) of the prismatic actuators greatly influences the size and shape of the workspaces. In Fig. 6, the five types of possible workspaces for a leg of PRRR configuration are shown. Each type of workspace depends on the stroke (D) of the prismatic actuators. It is possible to see that the addition of the

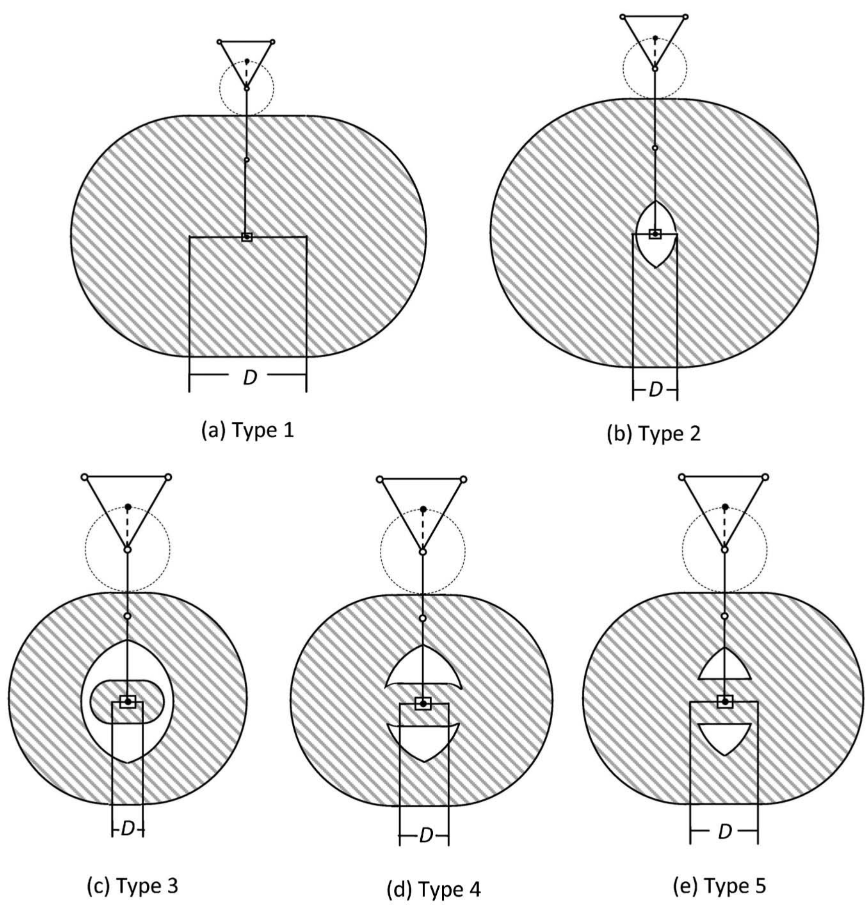


Fig. 6. Types of workspaces of the individual legs of a kinematically redundant planar parallel manipulator (PRRR).

prismatic actuator has the effect of stretching the workspace and reducing its non-dexterous area. This is where the advantage of kinematic redundancy becomes clear. If the prismatic actuator has a large enough stroke, the internal non-dexterous parts of the workspace vanish completely.

A workspace that does not contain holes (Type 1 shown in Fig. 6a) is possible regardless of the class of workspace of the individual RRR leg (Fig. 4). If the leg is of Class 1, the workspace, with the addition of kinematic redundancy, will be of Type 1 (Fig. 6a) with any value of D. If the leg is of Class 2 or 3, a prismatic actuator with a sufficiently large stroke can yield a Type 1 workspace for the leg. The following equations and Fig. 7(a) define and illustrate the conditions that must be met in order for the holes in the workspace to disappear for classes 2 and 3:

$$D \geq 2r_2 \tag{7}$$

$$D \geq 2\sqrt{r_2^2 - r_1^2} \tag{8}$$

where r_1 and r_2 are shown in Fig. 5. If the leg is of Class 2, Eq. (7) is to be met in order to cover the hole in the workspace. Equation (8) represents this same condition when a leg is of Class 3. For an identical value of r_2, a smaller stroke D is needed to obtain a Type 1 workspace when r_1 exists. As reflected by Eq. (8) and Fig. 7(a), the internal non-dexterous area vanishes when the height of the non-dexterous area (h in Fig. 7a) is shorter than the radius of the internal workspace (r_1). Note that Eq. (7) is the same as Eq. (8) with r_1 being zero.

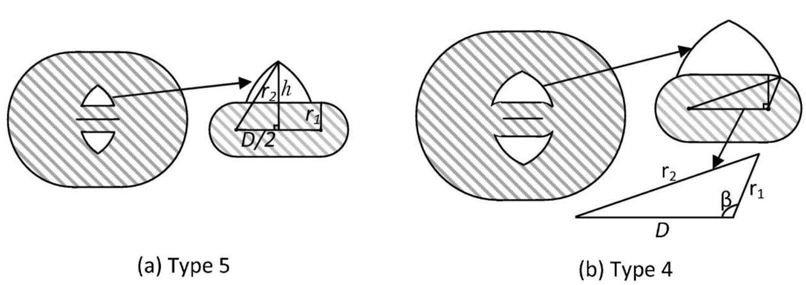
A Type 2 workspace is possible when the leg is of Class 2 and when the condition in Eq. (7) is not met (Fig. 6b).

The three remaining types of workspace are only possible with Class 3 legs. Type 3 is encountered when the stroke D is too short for the outer and inner workspaces to connect. This type, shown in Fig. 6(c), occurs when the prismatic actuator has a stroke respecting the following condition:

$$D < r_2 - r_1 \tag{9}$$

With Fig. 6(c), it is possible to see that when the left and right hand sides of Eq. (9) equate, the inner and external workspaces connect and become a Type 4 workspace.

Types 4 and 5 differ by the number of arcs or lines needed to define their internal non-dexterous workspaces (ten for Type 4 and six for Type 5). These last types are possible if the condition in Eq. (8) is not met. The differences between these types are caused by the stroke D of the prismatic actuator.



(a) Type 5 (b) Type 4

Fig. 7. Geometrical representation of the stroke limits for different workspace types.

Type 4, shown in Fig. 6(d), occurs when:

$$r_2 - r_1 \leq D < \sqrt{r_2^2 - r_1^2} \tag{10}$$

Examining Fig. 7(b), it can be seen that a Type 4 workspace becomes a Type 5 when the angle β is $\pi/2$ radians. When this happens, D is equal to $\sqrt{r_2^2 - r_1^2}$, thus yielding the right hand side of Eq. (10). Finally, Type 5 in Fig. 6(e) occurs when:

$$\sqrt{r_2^2 - r_1^2} \leq D < 2\sqrt{r_2^2 - r_1^2} \tag{11}$$

The left hand side of this equation is the same as the right hand side of Eq. (10) and as seen in Fig. 7(a), when h is shorter than the internal radius r_1, the hole in the workspace disappears and creates a Type 1 workspace. This happens when $D/2$ is equal or greater than $\sqrt{r_2^2 - r_1^2}$, thus yielding the right hand side of Eq. (11) as well as that of Eq. (8).

Distinguishing between Types 4 and 5 is important when determining the workspace in a general algorithm because the area is defined by a different number of lines and arcs.

4. FINAL WORKSPACE OF A 3-PRRR KINEMATICALLY REDUNDANT PARALLEL PLANAR MANIPULATOR

Once the workspace of each leg is established, the next step is to determine their intersection. Figure 8 shows an example of the workspace of one of the legs and the final workspace of a 3-PRRR planar manipulator. In this example, each leg is built with identical dimensions but with slightly different strokes. L_1, L_2 and L_3 for each leg are equal to 4, 3 and 2 units of length, respectively. The length of each side of the triangle defined by the coordinates (0,0), (1,2) and (2,0) as indicated in Fig. 8(b) corresponds to the stroke of each prismatic actuator. With these dimensions and strokes, each leg has a Type 4 workspace. Figure 8(a) shows the workspace of the leg corresponding to the horizontal prismatic actuator of Fig. 8(b). The workspace of the other two legs have different orientations due to the orientations of the prismatic actuators. Figure 8(b) shows the intersection of the workspaces of the three legs. The dotted lines in this figure indicate the boundaries of the inner workspace of each leg. Their intersection produces a triangular internal workspace for this example.

5. CONCLUSIONS

The workspace of each leg of a general geometry planar 3-RRR parallel manipulator has been established with the analogy of four-bar mechanisms. This was done by characterizing different categories of four-bar mechanisms and analyzing which categories permitted the end-effector to complete a full revolution for a given leg. A point is included in the workspace of a leg if the link representing the end-effector is categorized as a crank at that point.

Using the workspace of each leg of the 3-RRR, the influence of the redundant prismatic actuation was then added. From this, five possible types of workspaces have been identified and the inherent complexity of the problem became apparent as shown in Fig. 6.

The workspace of the manipulator consists of the intersection of the workspaces of all three legs. The method presented in this work does not require the prismatic actuators to form a

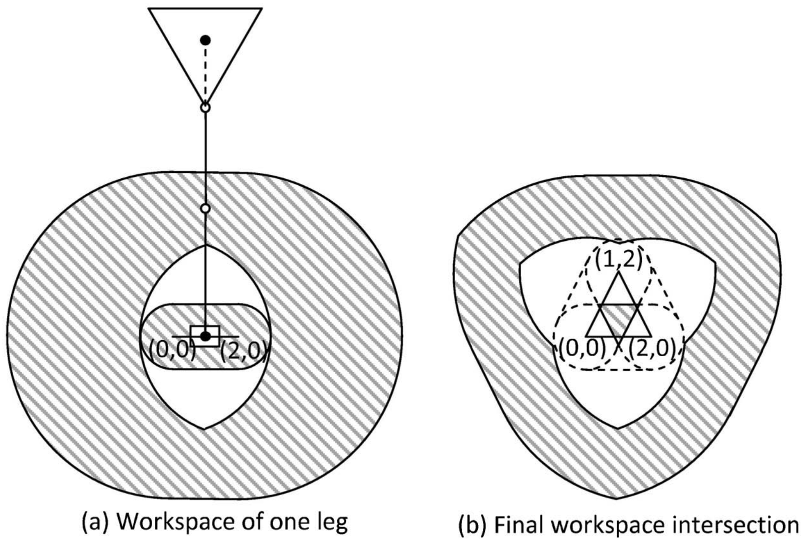


Fig. 8. Workspace of one of the PRRR legs and the final workspace of a 3-PRRR kinematically redundant planar parallel manipulator.

triangle. Furthermore, although the procedure was presented for a 3-PRRR manipulator, it should be noted that it can readily be extended to any n-PRRR planar manipulator. The manipulator workspace in such a case would consist of the intersection of the workspaces of the n legs, each obtained as presented here.

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