HAMILTONIAN AS ERROR INDICATOR IN THE *P*-VERSION OF FINITE ELEMENT METHOD

Yong-Lin Kuo[1], William L. Cleghorn[2], Kamran Behdinan[3]

[1] *Graduate Institute of Automation and Control, National Taiwan University of Science and Technology, Taipei City, Taiwan*

[2] *Department of Mechanical and Industrial Engineering, University of Toronto, Toronto, Ontario, Canada*

[3] *Department of Aerospace Engineering, Ryerson University, Toronto, Ontario, Canada*

E-mail: yl_kuo@yahoo.com

Received September 2009, Accepted May 2010

No. 09-CSME-56, E.I.C. Accession 3142

ABSTRACT

This paper presents the Hamiltonian-based error analysis applied to two-dimensional elasostatic problems. The accuracy enhancement is achieved by using the *p*-version of finite element method. The results show that the Hamiltonian error has faster rates of convergence at lower order of interpolation polynomials to compare with the energy error, and the Hamiltonian error clearly indicates great error reductions at a certain polynomial order. This can not only obtain an accurate enough solution but also save extra computational time. Another strategy is presented by computing the residual of the Hamiltonian-based governing equations. Relative values of residuals between elements can provide an index of selecting the best polynomial orders. Illustrative examples show the validities of the two approaches.

Keywords: finite element method; error indicator; *p*-version; hamiltonian.

L'HAMILTONIEN COMME INDICATEUR D'ERREUR DANS LA VERSION-P DE LA MÉTHODE DES ÉLÉMENTS FINIS

RÉSUMÉ

Cet article présente la méthode d'analyse d'erreur hamiltonienne appliquée à des problèmes élastostatiques à deux dimensions. L'amélioration de la précision est obtenue en utilisant la « p-version » de la méthode des éléments finis. Pour comparer avec l'erreur d'énergie, les résultats démontrent que la méthode hamiltonienne a des taux plus élevés de convergence pour des polynômes d'interpolation d'un ordre inférieur. La méthode d'analyse d'erreur hamiltonienne indique clairement des réductions d'erreur importantes à un certain degré de polynôme. Ceci peut non seulement produire une solution assez précise, mais aussi permet d'économiser le temps de calcul supplémentaire. Une autre stratégie est présentée qui utilise le calcul résiduel des équations hamiltoniennes de base. La valeur résiduelle relative entre les éléments peut fournir un indice de sélection du meilleur degré polynomial. Des exemples sont donnés pour illustrer la validité des deux approches.

Mots-clés : méthode des éléments finis; indicateur d'erreur; version-P; hamiltonien.

1. INTRODUCTION

There are the *h*- and *p*-versions of the finite element method to increase the level of accuracy. The *h*-version refines the mesh and uses fixed order polynomials. The *p*-version uses a fixed mesh, but increases the polynomial order. The *h*-version has been used for a long time [1–3]. The major advantage is that it is easy to implement. In contrast, the *p*-version has fast rates of convergence when the solutions are smooth [4]. Besides, the combination is called the *hp*-version [5, 6]. Shape functions vary as the number of nodes increases. This causes programming difficulties of the *p*-version. The hierarchic shape functions add new shape functions only when the number of nodes increases. A set of useful families of the hierarchical *p*-version were constructed [7, 8].

Error indicators are significant for evaluating accuracy of approximated solutions. An error indicator based on solution gradients, interpolation errors, and a posteriori error estimates were presented for the *r*-adaptive finite element method [9]. An anisotropic error indicator presented in the frame of the Laplace equation was extended to elliptic and parabolic problems [10]. A local error estimator corrected by a factor which depends only on the polynomial order of the element consists in an enhancement of an error indicator [11]. An energy-based error indicator consisting of modification of experimental measurements and numerical errors in the integration algorithm was presented [12]. The above energy-based error indicators were used in various applications. However, the indicators may not be very sensitive to errors when the problem domain is discretized as a large number of degrees of freedom.

Different from previous energy-based error indicator, Tabarrok *et al*. exploited the property that the Hamiltonian is invariant for some non-conservative systems [13]. This property of invariance can be utilized to define an error indicator for verifying numerical results. Stylinanou *et al*. provided the theory of Hamiltonian-based local error computations and the finite element applications on elasto-statics and elasto-dynamics [14]. A formula based on the residual of the Hamiltonian-based governing equation for a generalized conservation check was used to compute the error of finite element solutions [15]. Tabarrok *et al*. generalized the principle of least action by using the Hamiltonian [16]. This generalized principle can be formulated for the varying-domain systems that are dependent on the second derivatives. Based on the previous work, reference [17] presented two sets of generalized Hamiltonian formulations, matrix- and scalar forms, which were defined as error indicators. Although the paper provides basic forms of the Hamiltonians, their analytical expressions depend on the application problems. This paper demonstrated only Euler-Bernoulli beam problems solved by the *h*-refinement to show the validity of the Hamiltonians.

This paper utilizes the error indicators presented in [17] to address the following issues: (1) two-dimensional elasto-static problems are solved by the *p*-version; (2) the analytical expressions of the Hamiltonians for the cases of plane strain and plane stress are derived; (3) the Hamiltonian-based error indicator shows the sensitivities to errors. In contrast, the energy-based error indicator may not sense errors for high value of p when applying the *p*-version; (4) the Hamiltonian-based error indicator is suitable as an error index when applying adaptive *p*-version.

2. HAMILTONIAN-BASED ERROR INDICATORS

2.1. Residual of Hamiltonian-based governing equations

For two-dimensional elasto-static cases, the residuals in terms of the Hamiltonians are expressed as [14]

$$\alpha = \int_S \left(H_{11}n_x + H_{21}n_y\right)dS, \quad \beta = \int_S \left(H_{12}n_x + H_{22}n_y\right)dS \tag{1}$$

where S is the boundary of the domain, n_x and n_y are components of the unit normal vector outward to the boundary; the Hamiltonians terms H_{11}, H_{12}, H_{21}, H_{22} are given as

$$\begin{aligned} H_{11} &= -L + P_{11}u_x + P_{21}v_x, \quad & H_{21} &= P_{12}u_x + P_{22}v_x \\ H_{12} &= P_{11}u_y + P_{21}v_y, & H_{22} &= -L + P_{12}u_y + P_{22}v_y \end{aligned} \tag{2}$$

where L is the Lagrangian, u and v are displacements along the x and y axes, respectively, and $P_{11} = \partial L/\partial u_x$, $P_{12} = \partial L/\partial u_y$, $P_{21} = \partial L/\partial v_x$, and $P_{22} = \partial L/\partial v_y$.

An error indicator based on the residuals, Eqs. (1–2), is expressed as

$$\text{Error(Res)} = \sqrt{\left(\alpha^2 + \beta^2\right)/2} \tag{3}$$

Equation (1) is the integration of a set of Hamiltonian-based governing equations over the boundary of problem domain. If an exact solution is applied, the residual is zero. Thus, the residuals can be error indices for approximated solutions.

2.2. Error of Hamiltonians

An error indicator based on the error of Hamiltonians is written as

$$\text{Error(Ham)} = \sqrt{\sum_{i,j}^{1,2} \left[\int_\Omega \left(H_{ij}(FE) - H_{ij}(EX)\right)^2 \Big/ \left(H_{ij}(EX)\right)^2 dV\right]} \tag{4}$$

where Ω is the problem domain; $H_{ij}(FE)$ and $H_{ij}(EX)$ are the Hamiltonians of the finite element solutions and exact solutions, respectively; the Hamiltonians are functions of x and y.

2.3. Energy norm

The energy norm is introduced in order to be a benchmark, and it is defined as [4]

$$\text{Error(SE)} = \sqrt{|U_{EX} - U_{FE}|/U_{EX}} \tag{5}$$

where U_{EX} and U_{FE} are strain energies of the exact solutions and finite element solutions, respectively.

3. COMPARISON OF HAMILTONIAN AND ENERGY

3.1. Plane stress

The potential energy and the Hamiltonians are respectively expressed as [17]

$$U = \int_\Omega L dV, \quad H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \tag{6}$$

where $L = \frac{E}{2(1-v^2)}\left[u_x^2 + v_y^2 + 2vu_xv_y + (1-v)\left(u_y + v_x\right)^2\right]$; u and v are the displacements along the x- and y- directions, respectively; E is the Young's modulus;v is the poisson's ratio; V is the volume; the components of the Hamiltonians are

$$\left.\begin{aligned} H_{11} &= -L + \frac{E}{1-v^2}(u_x + vv_y)u_x + \frac{E}{2(1+v)}(u_y+v_x)v_x, \quad H_{12} = \frac{E}{1-v^2}(u_x+vv_y)u_y + \frac{E}{2(1+v)}(u_y+v_x)v_y \\ H_{21} &= \frac{E}{2(1+v)}(u_y+v_x)u_x + \frac{E}{1-v^2}(v_y+vu_x)v_x, \quad H_{22} = -L\frac{E}{2(1+v)}(u_y+v_x)u_y + \frac{E}{1-v^2}(y_y+vu_x)v_y \end{aligned}\right\} \tag{7}$$

3.2. Plane strain

Similarly, the potential energy and the components of the Hamiltonians are expressed as [17]

$$U = \int_\Omega L dV \tag{8}$$

$$\left.\begin{aligned} H_{11} &= -L + \frac{E}{(1+v)(1-2v)}\left[(1-v)u_x + vv_y\right]u_x + \frac{E}{2(1+v)}(u_y+v_x)v_x \\ H_{12} &= \frac{E}{(1+v)(1-2v)}\left[(1-v)u_x + vv_y\right]u_y + \frac{E}{2(1+v)}(u_y+v_x)v_y \\ H_{21} &= \frac{E}{2(1+v)}(u_y+v_x)u_x + \frac{E}{(1+v)(1-2v)}\left[vu_x + (1-v)v_y\right]v_x \\ H_{22} &= -L + \frac{E}{2(1+v)}(u_y+v_x)u_y + \frac{E}{(1+v)(1-2v)}\left[vu_x + (1-v)v_y\right]v_y \end{aligned}\right\} \tag{9}$$

where $L = \frac{E}{2(1+v)(1-2v)}\left[(1-v)\left(u_x^2+v_y^2\right) + 2vu_xv_y + \frac{1-2v}{2}(u_y+v_x)^2\right]$

To compare the Hamiltonians and strain energy, strain energy is constant, but the Hamiltonians, whose unit is Joul/m^3, are functions of independent variables x and y. Also, the Hamiltonians include quantities P_{ij}'s, called momentum terms. Thus, the Hamiltonians can be more sensitive according to the change of system momentum, and the error indicator defined in Eq. (4) can provide more information in error indices of approximate solutions.

4. ILLUSTRATIVE EXAMPLES

4.1. Unstressed Circular Hole in an Infinite Plate

A classical problem of the circular hole in an infinite plate subjected to unidirectional tension is considered and shown in Fig. 1 [4].

Two meshes of the plate are shown in Fig. 2. The p-version is used to enhance the accuracy of solution, and the value p, degree of the element shape function, is increased from 1 to 10. Fig. 3 shows that the three error indicators vs. the square root of the number of degrees of freedom for both meshes. The figure shows that the three errors converge monotonically. In contrast, the

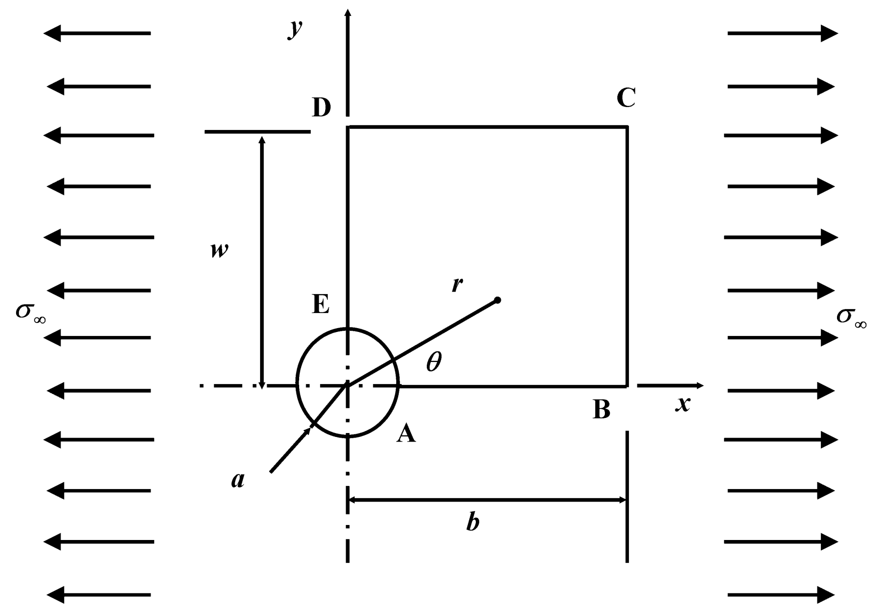

Fig. 1. A circular hole in an infinite plate subjected to tension σ_∞.

normal stresses at the circle boundary shown in Fig. 4 do not converge monotonically. The efficiency of the Hamiltonian-based approach is measured by reaching an expected accuracy which depends on the assigned value of p. For a practical consideration, it is important to know which p value may provide accurate enough solutions in order to save extra computations.

For the two-element mesh, Error(Ham) converges faster than Error(SE). This implies that Error(Ham) is more sensitive than Error(SE). The results show that the value of p greater than 5 produces relatively small reduction of errors based on Error(Ham). It is concluded that $p = 5$ provides the accurate enough solution. In contrast, the convergence rate of Error(SE) is like an exponential function, and it is difficult to predict which p value provides enough accuracy. Based on Fig. 4, the maximum x-directional normal stress occurs at node 3 (see Fig. 2). When the value of p is larger than 3, the stress error of node 3 is always below 5%. Thus, $p = 5$ provides a reasonable solution.

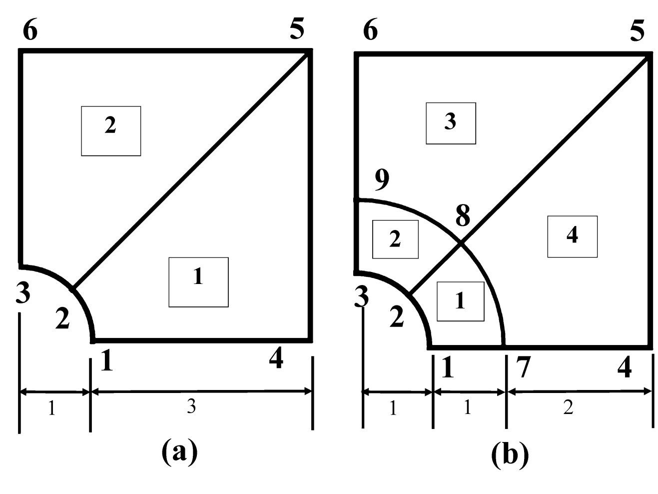

Fig. 2. Two types of meshes: (a) two-element mesh, (b) four-element mesh.

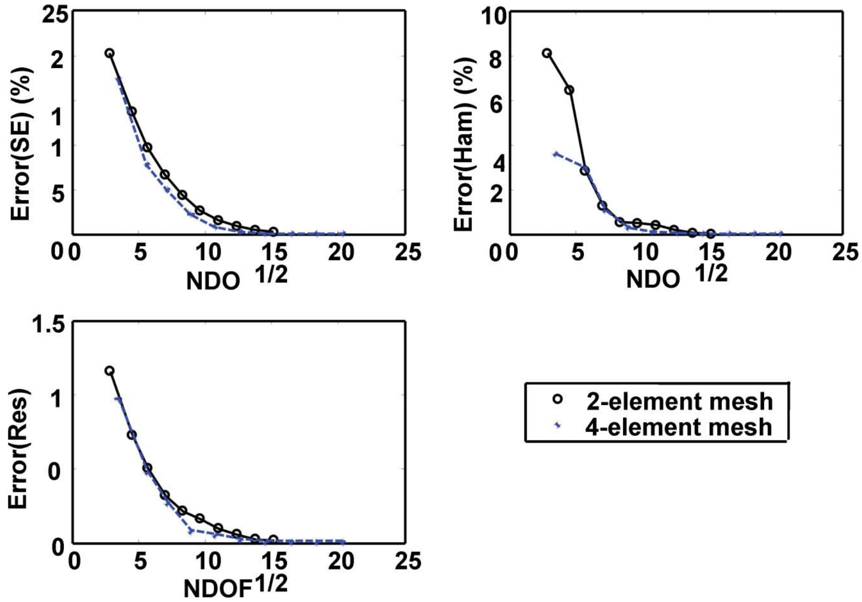

Fig. 3. Errors versus square root of the number of degrees of freedom.

For the four-element mesh, $p = 5$ may provide an accurate enough solution when examining Error(Ham). In contrast, through the examination of Error(SE), it is not easy to determine which value of p may provide accurate enough solutions. Besides, the stress error is 6.67% when $p = 4$. When p is greater than 4, the stress errors are always below 0.01%. Thus, $p = 5$ may provide an accurate normal stress. Although the four-element mesh provides higher accuracy for the same number of degrees of freedom, it is concluded that $p = 5$ is a sufficient value for both cases.

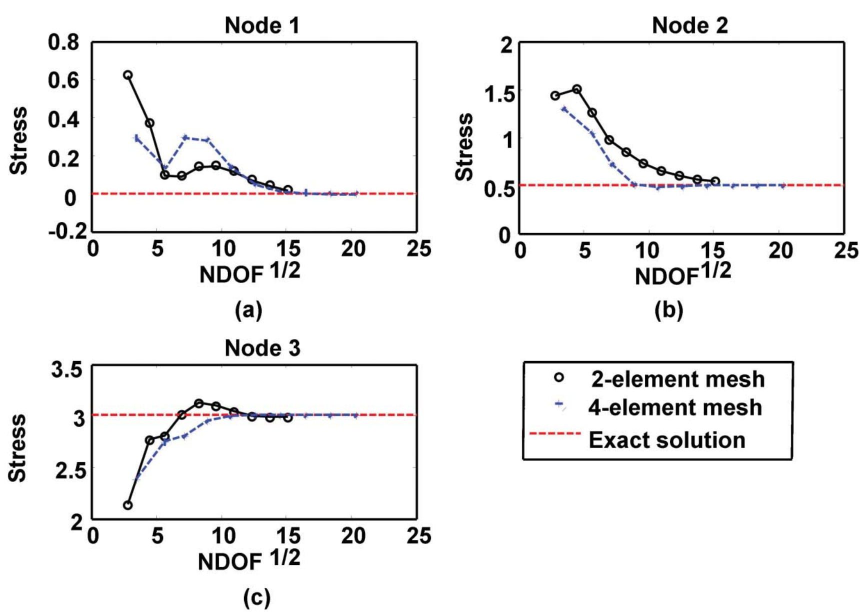

Fig. 4. The x-directional normal stress at the circle boundary for both meshes.

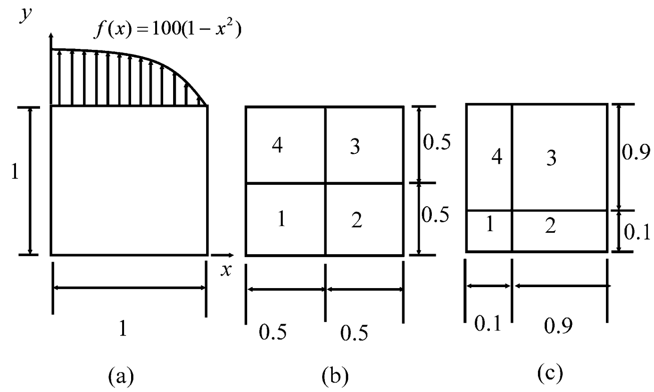

Fig. 5. Square panel under parabolic edge load, B.C. being $u(0,y)=0$ and $v(x,0)=0$: (a) square panel, (b) uniform mesh, (c) non-uniform mesh.

4.2. Square panel under parabolic edge load

A square panel under parabolic edge load shown in Fig. 5(a) is studied [18]. Three meshes are investigated as follows, and error analysis is compared to each other.

(1) *h*-version: The domain is divided into equal-size elements shown in Fig. 5(b), and each element has four nodes. Value *h*, size of elements, ranges from 1 (1 element) to 1/16 (256 elements).
(2) *p*-version: The equal-size mesh is shown in Fig. 5(b). The value *p* starts from 1 to 10 for each element.
(3) Adaptive *p*-version: The unequal-size mesh is shown in Fig. 5(c), and each element is assigned for different value *p*. The initial value *p* starts from 2 for each element, and then only the value of the element with the largest residual of the Hamiltonians is updated. It is expected that element 3 will require the highest value of *p* because of the largest area.

Figure 6 shows the error indicators vs. the square root of the degrees of freedom. To compare the *p*-version with the *h*-version, the *p*-version can provide higher convergence rate under the

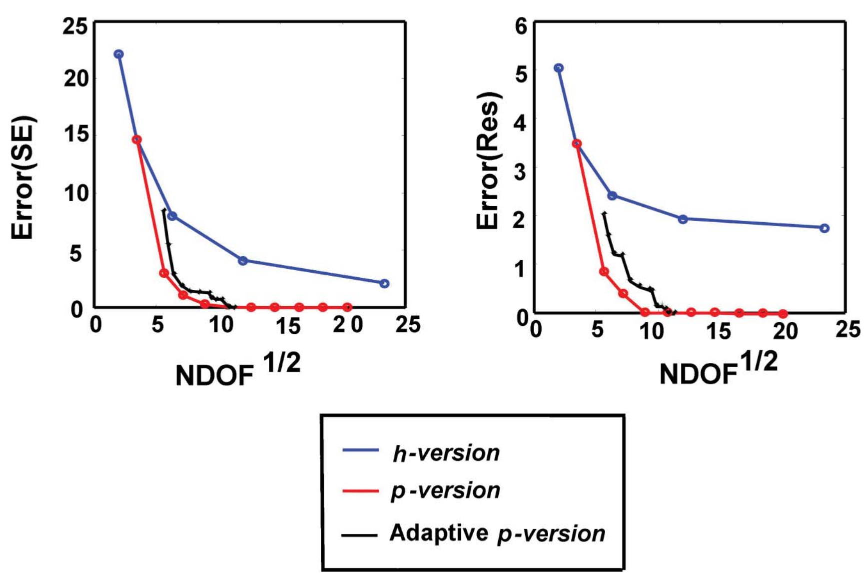

Fig. 6. Errors versus square root of the number of degrees of freedom.

Table 1. Adaptive p-version.

NDOF	Error(Res) / *p*				Total	Error(SE)
	Element 1	Element 2	Element 3	Element 4	Error(Res)	(%)
32	0.1025 /2	0.1186 /2	1.3561 /2	0.4577 /2	2.0349	8.4778
36	0.0883 /2	0.0854 /2	1.0582 /3	0.3746 /2	1.6065	5.5421
42	0.0385 /2	0.0722 /2	0.8894 /4	0.2305 /2	1.2306	2.9757
50	0.0361 /2	0.0704 /2	0.8741 /5	0.2126 /2	1.1932	1.9361
60	0.0303 /2	0.0628 /2	0.3892 /6	0.1943 /2	0.6766	1.4499
72	0.0284 /2	0.0604 /2	0.2756 /7	0.1925 /2	0.5569	1.3708
86	0.0263 /2	0.0586 /2	0.2201 /8	0.1885 /2	0.4935	1.3469
91	0.0252 /2	0.0541 /2	0.2117 /8	0.1772 /3	0.4682	0.8589
98	0.0125 /2	0.0537 /2	0.0360 /8	0.0540 /4	0.1562	0.7582
107	0.0124 /2	0.0536 /2	0.0359 /8	0.0533 /5	0.1552	0.7556
112	0.0061 /2	0.0366 /3	0.0253 /8	0.0290 /5	0.0970	0.3960
119	0.0003 /2	0.0042 /4	0.0013 /8	0.0008 /5	0.0066	0.0894
128	0.0000 /2	0.0040 /5	0.0011 /8	0.0007 /5	0.0058	0.0700

same number of degrees of freedom. In this example, $p = 4$ is enough, since the residual is close to zero when $p > 3$. For adaptive *p*-version (see Table 1), the optimal values of p are 2, 5, 8 and 5 for elements 1 through 4, respectively, which are consistent with Ref [18]. Thus, Error(Res) is suitable for the adaptive *p*-version.

5. CONCLUSIONS

The paper presents the application of Hamiltonian-based error indicators to two-dimensional elasto-static problems solved by the *p*-version. The indicators are based on the error of the Hamiltonians, Error(Ham), and the residual of the Hamiltonian-based governing equations, Error(Res). Also, the formulations of the Hamiltonians for plane stress and plane strain are presented and compared with strain energy. The errors are compared with the error of strain energy, Error(SE). Since it converges smoothly, it is difficult to determine a proper value p or h. In contrast, Error(Ham) converged faster and its error reduction is clearly examined, so it is easier to determine a reasonable value. Since Error(Res) can be applied in instances where the exact solution is not available, it is suitable for the adaptive mesh.

REFERENCES

1. Zienkiewicz, O.C. and Tayler, R.L., *Finite Element Method*, McGraw Hill, New York, 1991.
2. Bathe, K., *Finite Element Procedures*, Prentice Hall, Englewood, 1996.
3. Lweis, P.E. and Ward, J.P., *The Finite Element Method: Principles and Applications*, Addison-Wesley, 1991.
4. Szabo, B.A. and Babuska, I., *Finite Element Analysis*, Wiley, New York, 1991.
5. Babuska, I. and Suri, M., "The *p*- and *h-p* versions of the finite element method, an overview," *Computer Methods in Applied Mechanics and Engineering*, Vol. 80, pp. 5–26, 1990.

6. Babuska, I. and Suri, M., "The *p* and *h-p* versions of the finite element method, basic principles and properties," *SIAM Review*, Vol. 36, No. 4, pp. 578–632, Dec. 1994.
7. Peano, A.G., "Hierachies of conforming finite elements for plane elasticity and plate bending," *Computers & Mathematics with Applications*, Vol. 2, pp. 211–224, 1976.
8. Peano, A.G., "Conforming approximations for Kirchhoff plates and shells," *International Journal for Numerical Methods in Engineering*, Vol. 14, pp. 1273–1291, 1979.
9. Cao, W., Huang, W. and Russell, R.D., "Comparison of two-dimensional *r*-adaptive finite element methods using various error indicators," *Mathematics and Computers in Simulation*, Vol. 56, No. 2, pp. 127–143, 2001.
10. Picasso, M., "An anisotropic error indicator based on Zienkiewicz-Zhu error estimator: Application to elliptic and parabolic problems," *SIAM Journal of Scientific Computing*, Vol. 24, No. 4, pp. 1328–1355, 2003.
11. Tarancón, J.E., Fuenmayor, F.J. and Baeza, L., "An a posteriori error estimator for the *p*- and *hp*-versions of the finite element method," *International Journal for Numerical Methods in Engineering*, Vol. 62, No. 1, pp. 1–18, 2005.
12. Ahmadizadeh, M. and Mosqueda, G., "Online energy-based error indicator for the assessment of numerical and experimental errors in a hybrid simulation," *Engineering Structures*, Vol. 31, No. 9, pp. 1987–1996, 2009.
13. Tabarrok, B., Tezer, C. and Stylianou, M., "A note on conservation principles in classical mechanics," *Acta Mechanica*, Vol. 107, pp. 137–152, 1994.
14. Stylianou, M.C. and Tabarrok, B., "Consistent local error computations, Part I: Theory and Part II: F.E. applications," *International Journal for Numerical Methods in Engineering*, Vol. 39, pp. 3219–3246, 1996.
15. Tabarrok, B., Qin, Z. and Stylianou, M., "Generalized conservation checks in finite element analysis," *International Journal for Numerical Methods in Engineering*, Vol. 43, pp. 1295–1307, 1998.
16. Tabarrok, B. and Cleghorn, W.L., "Application of principle of least action to beam problems," *Acta Meachanica*, Vol. 142, No. 1–4, pp. 235–243, 2000.
17. Kuo, Y.L., Behdinan, K. and Cleghorn, W.L., "Hamiltonian-based error computations," *Communications in Numerical Methods in Engineering*, Vol. 22, No. 4, pp. 283–299, 2006.
18. Bertoti, E. and Szabo, B., "Adaptive selection of polynomial degrees on a finite element mesh," *International Journal for Numerical Methods in Engineering*, Vol. 42, 561–578, 1998.