NUMERICAL ANALYSIS OF DISPLACEMENTS IN SPATIAL MECHANISMS WITH SPHERICAL JOINTS UTILIZING AN EXTENDED D-H NOTATION

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ABSTRACT

Spherical joints consist of a pair of concave and convex spherical surfaces engaged in such a way as to prevent translational motion of the ball and socket whilst simultaneously allowing three degrees of rotational freedom. The kinematics of spatial mechanisms comprising links and joints are commonly analyzed using the Denavit-Hartenberg (D-H) notation. However, whilst this method allows the kinematics of mechanisms containing prismatic, revolute, helical and cylindrical joints to be explicitly defined, it cannot be directly applied to mechanical systems containing spherical pairs. Accordingly, this paper proposes an extended D-H notation which allows the independent parameters of any spatial mechanism, including one with spherical pairs, to be derived for analysis and synthesis purposes. The validity of the proposed notation is demonstrated via its application to the analysis of mechanisms containing revolute (R), spherical (S), cylindrical (C) and prismatic (P) joints. The results confirm the viability of the extended D-H notation as a means of analyzing the displacements of mechanical systems containing kinematic chains such as RSCR, RSCP, CSSR and CSSP.

Keywords: spatial mechanism; denavit-hartenberg notation; displacement analysis.

ANALYSE NUMÉRIQUE DES DÉPLACEMENTS DE MÉCANISMES SPATIAUX À ARTICULATIONS SPHÉRIQUES UTILISANT LA NOTATION D-H ÉTENDUE

RÉSUMÉ

Les articulations sphériques consistent en une paire de surfaces sphériques concave et convexe engagées de façon à empêcher un mouvement de translation de la rotule et de la connection tout en allouant simultanément trois degrés de liberté. La cinématique de mécanismes spatiaux comprenant des liaisons et articulations sont couramment analysées au moyen de la notation Denavit-Hartenberg (D-H). Toutefois, alors que cette méthode permet la définition claire des mécanismes aux articulations prismatiques, rotoïdes, hélicoïdales et cylindriques, elle ne peut être appliquée directement à des systèmes à paires sphériques. Par conséquent, nous proposons dans cet article une notation D-H étendue qui permet aux paramètres indépendants de n'importe lequel des mécanismes spatiaux, incluant ceux de paires sphériques, d'être dérivées pour fin d'analyse et de synthèse. La validité de la notation proposée est démontrée par l'application à l'analyse de mécanismes à articulations rotoïdes (R), sphériques (S), cylindriques (C) et prismatiques (P). Les résultats confirment la viabilité de la notation D-H étendue comme moyen d'analyser les déplacements de systèmes mécaniques contenant des chaînes cinématiques tel que RSCR, RSCP, CSSR et CSSP.

Mots-clés : mécanisme spatial; notation Denavit-Hartenberg; analyse de déplacement.

1. INTRODUCTION

Generally speaking, mechanical systems can be decomposed into a mechanism arrangement of interconnected links and joints. When designing such systems, it is necessary to analyze the relative trajectories and velocities of the various components in order to ensure that the mechanism provides the degrees of freedom required to accomplish the designated task. This type of analysis, referred to kinematics, conventionally classifies the joints within a system as either prismatic (P), revolute (R), helical (H), cylindrical (C) or spherical (S). Whilst the analysis of planar (2D) mechanisms is relatively straightforward, that of spatial mechanisms is far more challenging due to the geometric complexity of the pairs' characteristics in 3D space.

To resolve this problem, various systematic techniques have been proposed for modeling generic 2D and 3D mechanisms. Of these techniques, the Denavit-Hartenberg (D-H) notation is one of the most commonly applied [1]. In the D-H modeling approach, the shape and motion of any link with respect to its connected link is described in terms of the link angle, the link offset, the link length and the twist angle. The D-H notation enables a simple mathematical representation of a link to be derived, and can be applied to all mechanical systems comprising prismatic, revolute, helical or cylindrical joints. However, it cannot be directly applied to the kinematic analysis of mechanisms comprising spherical pairs since for this type of joint, the relative motion is not restricted to a single axis. Accordingly, various researchers have proposed specific methods for the mobility analysis of spatial mechanisms with spherical joints. For example, Sheth and Uicker [2] established a generalized symbolic notation for a spherical joint based on the kinematically-equivalent combination of three revolute pairs with mutually-orthogonal characteristic axes. Sandor and Erdman [3] analyzed the rotation and translation of spherical joints using a 4×4 homogenous notation. Yang [4] analyzed the displacement of spatial five-link mechanisms utilizing 3×3 matrices with dual-number elements. Fischer [5] used derivative-operator matrices to analyze the displacement of spatial mechanisms with spherical joints. Gupta [6] developed a displacement equation to analyze the mobility region of PRSPR kinematic chains using a kinematic loop-closure equation. Liu and Cheng [7] proposed a two-step procedure for determining the singular positions of a 3RPS parallel manipulator based on the assumption that the workspace of the moving platform was represented as an inclined solid cylinder on the coordinate system defined by the heights of the three spherical joints. Akcali and Multu [8] presented a method for solving the direct kinematics problem of a Steward platform by modeling the 3D problem as eight 2D problems on different planes and solving the three fundamental equations involving the three angles between the base and the 2D planes in order to obtain the corresponding 16 solution sets. Ting and Zhu [9] presented a closed-form equation describing the spherical trajectory of the ball in the socket of a spherical joint and showed that the equation enabled the classification of spherical joints in accordance with the required ball rotatability and socket opening.

However, the studies above present only the position-governing equation of the spherical joint, i.e., the orientation-governing equations of the links attached to the spherical joint are not considered. As a result, the proposed methods do not satisfy the necessary matrix property for links forming a closed loop; namely that the concatenation of all the transformation matrices forms an identity matrix. Accordingly, the present study proposes an extended D-H notation to support the kinematic modeling of all spatial mechanisms, including those containing spherical joints. Section 2 analyzes four generic links, namely P-R, R-S, S-R and S-S, and derives the corresponding coordinate assignment rules and link parameters. In Section 3, the validity of the proposed modeling approach is demonstrated by deriving the governing equations for the links within generic RSCR, RSCP, CSSP and RSSP mechanisms and using these equations to analyze the corresponding input-output relationship. Finally, Section 4 presents some brief conclusions.

Note that in the analysis performed in this study, the position vector $r_x\boldsymbol{i}+r_y\boldsymbol{j}+r_z\boldsymbol{k}$ is written in the form of the column vector ${}^j\boldsymbol{r}=[\mathrm{r_x}\ \mathrm{r_y}\ \mathrm{r_z}\ 1]^{\mathrm{T}}$, where the pre-superscript "j" of the leading symbol ${}^j\boldsymbol{r}$ indicates that the vector is referred with respect to coordinate frame $(\mathrm{xyz})_{\mathrm{j}}$. Furthermore, given the position vector of a point ${}^j\boldsymbol{r}$, its transformation ${}^k\boldsymbol{r}$ is represented by the matrix product ${}^k\boldsymbol{r}={}^k\boldsymbol{A}_j\,{}^j\boldsymbol{r}$, where ${}^k\boldsymbol{A}_j$ is a 4×4 matrix defining the position and orientation (i.e., pose) of an arbitrary frame $(\mathrm{xyz})_j$ with respect to another frame $(\mathrm{xyz})_k$. As shown in [10,11], ${}^k\boldsymbol{A}_j$ has the form

$$
{}^k\mathbf{A}_j=\begin{bmatrix} I_x & J_x & K_x & P_x \\ I_y & J_y & K_y & P_y \\ I_z & J_z & K_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{1}
$$

The inverse of ${}^k\boldsymbol{A}_j$ can be expressed as

$$
{}^k\mathbf{A}_j^{-1}={}^j\mathbf{A}_k=\begin{bmatrix} I_x & I_y & I_z & -(I_xP_x+I_yP_y+I_zP_z) \\ J_x & J_y & J_z & -(J_xP_x+J_yP_y+J_zP_z) \\ K_x & K_y & K_z & -(K_xP_x+K_yP_y+K_zP_z) \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2}
$$

The notation rules described above are equally applicable to the unit directional vector ${}^j\boldsymbol{n}=[\mathrm{n}_x\ \mathrm{n}_y\ \mathrm{n}_z\ 0\ ]^{\mathrm{T}}$. Note that for the case where the vector is referred to the world coordinate frame $(\mathrm{xyz})_0$, the pre-superscript "0" is deliberately omitted for reasons of convenience.

2. MODELING LINKS AND JOINTS

When analyzing a mechanism containing spherical joints, the matrix used to model the link terminating at the spherical joint can be embedded within a concatenated matrix comprising the transformation matrices of all the other links and joints in the mechanism irrespective of their type. In general mechanical systems, revolute (R), prismatic (P) and helical (H) joints have a single degree of freedom (denoted henceforth as f=1) and allow for either rotation or translation (R and H), or dependent rotation and translation (H). For a cylindrical (C) joint, two independent motions (i.e., rotational and translational) are permissible, and thus f=2. Finally, spherical pairs permit roll, pitch and yaw motions, i.e., f=3. Significantly, cylindrical joints can be modeled using the same D-H notation as that used for revolute joints, and consequently the links within a mechanical system can be classified as belonging to one of four different types depending upon the nature of the pair elements at either end of the link, namely P-R, R-S, S-R and S-S, respectively. The parameters and pose matrices of each type of link are described in the sections below.

(A) P-R link

A P-R link has a prismatic joint at one end and a revolute joint at the other. As shown in Fig. 1, the link is characterized by two parameters, namely the link length a_i and the twist angle

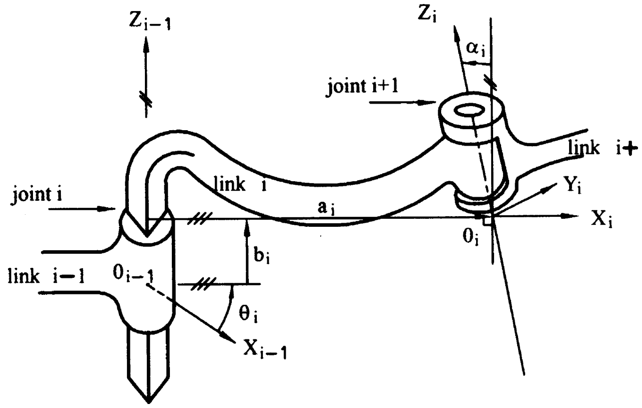

Fig. 1. Illustration of P-R link.

α_i, respectively, where a_i is the length of the common normal between the axes of the two joints and α_i is the angle between the two axes of the joints measured on a plane perpendicular to their common normal. The homogenous transformation matrix of frame $(xyz)_i$ with respect to frame $(xyz)_{i-1}$ can be written as

$$
\begin{aligned}
{}^{i-1}\mathbf{A}_i &= \mathbf{Rot}(z, \theta_i)\mathbf{Trans}(0,0\ b_i)\ \mathbf{Trans}(a_i,0,0)\ \mathbf{Rot}(x, \alpha_i) \\
&= \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & b_i \\ 0 & 0 & 0 & 1 \end{bmatrix},
\end{aligned} \tag{3}
$$

where $C\theta_i$ and $S\theta_i$ denote cosine θ_i and sine θ_i, respectively. Note that in Eq. (3), θ_i is the angle between axes x_{i-1} and x_i measured on the plane normal to the axis of joint i, and b_i is the distance between axes x_{i-1} and x_i measured along the axis of joint i. In this particular link, θ_i and b_i are pair variables if joint i is a cylindrical joint.

(B) R-S link

Figure 2 shows a typical R-S link comprising a revolute joint at one end and a spherical joint at the other. The link is characterized by the link length a_i, the axis of joint i and the center point 0_i of spherical joint i+1. The link length a_i is the normal distance between the center points of

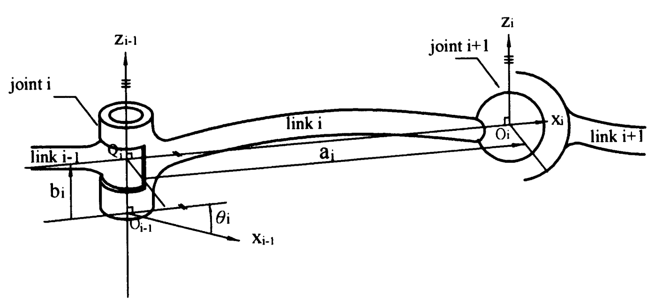

Fig. 2. Illustration of R-S link.

the two joints. The frame associated with link i is mounted at 0_i, and is positioned such that the z_i axis is parallel to the axis of joint i and passes through 0_i. Meanwhile, the x_i axis is aligned with the line from Q_i to 0_i. Therefore, the homogenous transformation matrix of frame $(xyz)_i$ with respect to frame $(xyz)_{i-1}$ can be written as

$$^{i-1}\mathbf{A}_i = \mathbf{Trans}(0,0,b_i)\ \mathbf{Rot}(z,\theta_i)\mathbf{Trans}(a_i,0,0) = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_iC\theta_i \\ S\theta_i & C\theta_i & 0 & a_iS\theta_i \\ 0 & 0 & 1 & b_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

As in the P-R link, θ_i is the angle between axes x_{i-1} and x_i and is measured on the plane normal to the axis of joint i, while b_i is the distance between axes x_{i-1} and x_i and is measured along the axis of joint i. Note that θ_i and b_i are both pair variables if the joint is a cylindrical pair, while θ_i is the only pair variable if the joint is a revolute joint.

(C) S-R link

Figure 3 presents a schematic illustration of a S-R link, in which joint i is spherical and joint i+1 is revolute. Although the geometry of the S-R link is very similar to that of the R-S link, the modeling method is entirely different. The S-R link is characterized by the link length a_i, the axis of joint i+1 and the center point 0_{i-1} of spherical joint i. The link length, a_i, is the normal distance between the center points of the two joints. The frame associated with link i, i.e., $(xyz)_i$, is mounted at 0_i and is orientated such that the z_i axis is aligned with the axis of joint i+1, while the x_i axis is aligned with the perpendicular from point 0_{i-1} to the axis of joint i+1. The spherical joint completely suppresses any linear motion between neighboring links, i.e., it limits the possible motion to rotation only. The relative orientation matrix is therefore given by

$$\mathbf{RPY}(\phi_i,\psi_i,\Gamma_i) = \mathbf{Rot}(z,\phi_i)\ \mathbf{Rot}(y,\psi_i)\ \mathbf{Rot}(x,\Gamma_i)$$

$$= \begin{bmatrix} C\phi_iC\psi_i & C\phi_iS\psi_iS\Gamma_i - S\phi_iC\Gamma_i & C\phi_iS\psi_iC\Gamma_i + S\phi_iS\Gamma_i & 0 \\ S\phi_iC\psi_i & S\phi_iS\psi_iS\Gamma_i + C\phi_iC\Gamma_i & S\phi_iS\psi_iC\Gamma_i - C\phi_iS\Gamma_i & 0 \\ -S\psi_i & C\psi_iS\Gamma_i & C\psi_iC\Gamma_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I_x & J_x & K_x & P_x \\ I_y & J_y & K_y & P_y \\ I_z & J_z & K_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

The homogenous transformation matrix of frame $(xyz)_i$ with respect to frame $(xyz)_{i-1}$ can then be written as

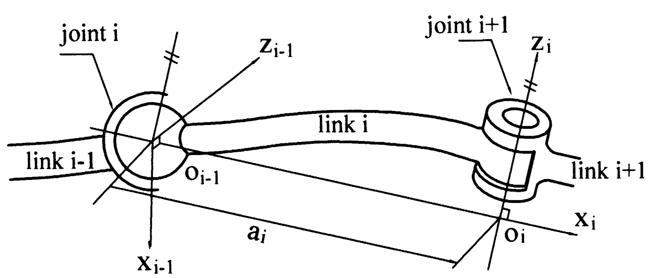

Fig. 3. Illustration of S-R link.

$$^{i-1}\mathbf{A}_i = \mathbf{RPY}(\phi_i,\psi_i,\Gamma_i)\mathbf{Trans}(a_i,0,0)$$

$$= \begin{bmatrix} C\phi_i C\psi_i & C\phi_i S\psi_i S\Gamma_i - S\phi_i C\Gamma_i & C\phi_i S\psi_i C\Gamma_i + S\phi_i S\Gamma_i & a_i C\phi_i C\psi_i \\ S\phi_i C\psi_i & S\phi_i S\psi_i S\Gamma_i + C\phi_i C\Gamma_i & S\phi_i S\psi_i C\Gamma_i - C\phi_i S\Gamma_i & a_i S\phi_i C\psi_i \\ -S\psi_i & C\psi_i S\Gamma_i & C\psi_i C\Gamma_i & -a_i S\psi_i \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6}$$

Note that since any S-R link *i* is connected to link *i*-1 via a spherical joint with f=3, then the joint variables are ϕ_i, ψ_i and Γ_i , respectively.

From Eq. (5), the rotations ϕ_i, ψ_i and Γ_i about the three mutually-perpendicular axes are given as

$$\phi_i = \tan^{-1}\left(I_y/I_x\right), \quad 0^0 \le \phi_i \le 360^0 \tag{7}$$

$$\psi_i = \sin^{-1}(-I_z), \quad -90^0 \le \psi_i \le 90^0 \tag{8}$$

$$\Gamma_i = \tan^{-1}\left(J_z/K_z\right). \quad 0^0 \le \Gamma_i \le 360^0. \tag{9}$$

Note that Eqs. (7) and (9) hold so long as $C\psi_i$ is not equal to zero. In the event that $C\psi_i$ =0, angles ϕ_i and Γ_i can be obtained via Newton iteration by equating the corresponding elements in the matrix given in Eq. (5). Moreover, when performing the numerical calculations using FORTAN software, the inverse of the sine function in Eq. (8) is single-valued, and is thus easily derived. Ω_{i-1}

A spherical joint comprises a solid ball and a hollow socket. The spherical coordinates include the socket radius *R*, the azimuth angle ϕ_i (measured in the $x_{i-1}y_{i-1}$ plane from the x_{i-1} axis and having a value of $0^\circ \leqq \phi_i \leqq 360^\circ$, see Eq. (7)) and the polar angle ψ_i ($-90^\circ \leqq \psi_i \leqq 90^\circ$, see Eq. (8)). Connecting all the coordinates (R_i, ψ_i, ϕ_i) obtained from the working space of the mechanism yields a contour Σ_{i-1} on the socket surface, where Σ_{i-1} is either open or closed, depending on whether the input motion is limited or not.

Consider the relative pose of these assigned frames $(xyz)_i$. Coordinates R_i, ϕ_i and ψ_i describe the region bounded by locus Σ_{i-1} traced by x_i rather than the interference-free region Ω_{i-1} of the socket. Therefore, the link shape must be taken into account if shape compensation is not applied when determining Ω_{i-1}. That is, the region will be dimensionally incorrect if the link shape is not considered. In most cases, link shape-compensation can be performed simply by offsetting the link radius outwardly along Σ_{i-1} provided that link *i* is axis-symmetrical about the x_i axis.

(D) S-S link

Figure 4 illustrates a S-S link with a spherical joint at either end. The link is characterized by the link length a_i and the center points 0_{i-1} and 0_i of the two spherical joints. The link length a_i is simply the distance between 0_{i-1} and 0_i. The origin of the link frame, $(xyz)_i$, is set at 0_i, i.e., the center of joint *i*+1, and is orientated such that the x_i axis is aligned with the line passing through the two centers of the spherical joints. Meanwhile, the z_i axis is perpendicular to the x_i

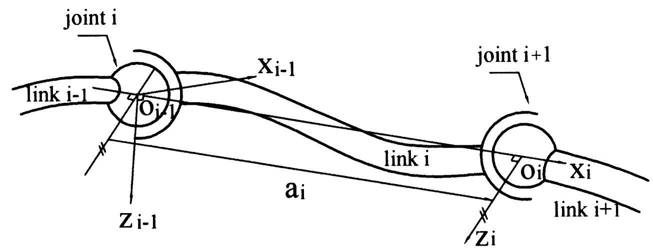

Fig. 4. Illustration of S-S link.

axis and passes through 0_i. Therefore, the homogenous transformation matrix of frame $(xyz)_i$ with respect to frame $(xyz)_{i-1}$ is given by

$$
\begin{aligned}
{}^{i-1}\mathbf{A}_i &= \mathbf{RPY}(\phi_i,\psi_i,\Gamma_i)\mathbf{Trans}(a_i,0,0) \\
&= \begin{bmatrix} C\phi_i C\psi_i & C\phi_i S\psi_i S\Gamma_i - S\phi_i C\Gamma_i & C\phi_i S\psi_i C\Gamma_i + S\phi_i S\Gamma_i & a_i C\phi_i C\psi_i \\ S\phi_i C\psi_i & S\phi_i S\psi_i S\Gamma_i + C\phi_i C\Gamma_i & S\phi_i S\psi_i C\Gamma_i - C\phi_i S\Gamma_i & a_i S\phi_i C\psi_i \\ -S\psi_i & C\psi_i S\Gamma_i & C\psi_i C\Gamma_i & -a_i S\psi_i \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned} \tag{10}
$$

Although, an infinite number of possibilities exist for the direction of z_i, an S-S link spins about the x_i axis, and thus the relative poses of the neighboring links are unchanged. Therefore, for analytical convenience, the spin angle Γ_i can be specified simply as zero (i.e., the z_i axis is parallel to the z_{i-1} axis), with the result that ϕ_i and ψ_i are the joint variables.

3. ILLUSTRATIVE EXAMPLES

In order to describe the mathematical relationship between the various links of a mechanism with n links, it is first necessary to label the individual links sequentially starting from the base of the mechanism (marked as "0"). A coordinate frame $(xyz)_i$ can then be assigned to each link i in accordance with the extended D-H notation presented in Section 2. In general, the homogenous transformation matrix of the k^{th} frame, i.e., $(xyz)_k$, with respect to the base frame, $(xyz)_0$, is given by

$$ {}^{0}\mathbf{A}_k = \prod_{i=1}^{k} {}^{i-1}\mathbf{A}_i. \tag{11} $$

If the links within the mechanism form a closed loop, the concatenation of the transformation matrices associated with the various links forms an identity matrix, i.e.,

$$ \prod_{i=1}^{n+1} {}^{i-1}\mathbf{A}_i = \mathbf{I}. \tag{12} $$

Equation (12) enables the kinematic equations of any spatial mechanism to be derived. Meanwhile, the pose of any link k within the mechanism can be determined from Eq. (11). Furthermore, the locus of the position vector of any point ${}^{k}\boldsymbol{r}=[r_x\ r_y\ r_z\ 1]^T$ on link k can be derived from the relationship ${}^{0}\boldsymbol{r}={}^{0}\boldsymbol{A}_k\,{}^{k}\boldsymbol{r}$. The corresponding velocity and acceleration can then be obtained via differentiation.

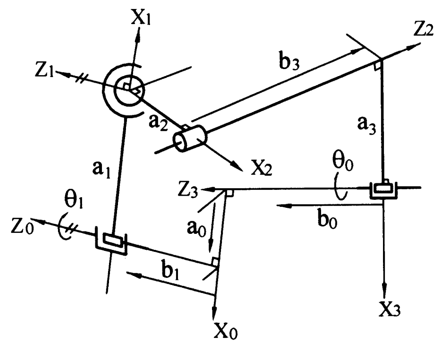

Fig. 5. Illustration of RSCR mechanism.

In the following sections, the validity of the proposed modeling methodology is demonstrated via its application to mechanisms containing generic RSCR, RSCP, CSSP and RSSP chains, respectively.

(A) RSCR mechanism

According to the Kutzbach criterion [12], the RSCR mechanism (see Fig. 5) has a single degree of freedom, i.e., f=1. The kinematic parameters for each of the four links in the RSCR mechanism are summarized in Table 1. Note that in this table, a_1, a_2, a_3, a_0, b_1, b_0, α_3 and α_0 are link parameters; θ_1, θ_0, Γ_2, ψ_2, ϕ_2, θ_3 and b_3 are joint variables, and θ_1 and θ_0 are the input and output displacements, respectively,

The matrix given in Eq. (12) is computationally complex. However, its solution can be simplified by pre-multiplying both sides of the equation by $({}^0\boldsymbol{A}_1{}^1\boldsymbol{A}_2)^{-1}$ and then post-multiplying the two sides of the resulting equation by $({}^0\boldsymbol{A}_1)$ to separate the position equation (i.e., the 4th column of the matrix) from the three rotation motions (i.e., $RPY(\phi_i,\psi_i,\Gamma_i)$). By doing so, it can be shown that

$$
{}^2\mathbf{A}_3{}^3\mathbf{A}_0{}^0\mathbf{A}_1 = \begin{bmatrix} I_x & J_x & K_x & P_x \\ I_y & J_y & K_y & P_y \\ I_z & J_z & K_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left({}^1\mathbf{A}_2\right)^{-1} = {}^2\mathbf{A}_1
$$

$$
= \begin{bmatrix} C\phi_2 C\psi_2 & S\phi_2 C\psi_2 & -S\psi_2 & -a_2 \\ C\phi_2 S\psi_2 S\Gamma_2 - S\phi_2 C\Gamma_2 & S\phi_2 S\psi_2 S\Gamma_2 + C\phi_2 C\Gamma_2 & C\psi_2 S\Gamma_2 & 0 \\ C\phi_2 S\psi_2 C\Gamma_2 + S\phi_2 S\Gamma_2 & S\phi_2 S\psi_2 C\Gamma_2 - C\phi_2 S\Gamma_2 & C\psi_2 C\Gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{13}
$$

Table 1. Kinematic parameters of RSCR mechanism

Parameter	Link 1	Link 2	Link 3	Link 0
Link type	R-S	S-C	C-R	R-R
Distance b_i	b_1		b_3	b_0
Angle θ_i	θ_1		θ_3	θ_0
Length a_i	a_1	a_2	a_3	a_0
Twist angle α_i	0		α_3	α_0
Yaw Γ_i		Γ_2		
Pitch ψ_i		ψ_2		
Roll ϕ_i		ϕ_2		

where

$$I_x = (C\theta_0 C\theta_3 - S\theta_0 S\theta_3 C\alpha_3)C\theta_1 + (-S\theta_0 C\theta_3 C\alpha_0 - C\theta_0 S\theta_3 C\alpha_0 C\alpha_3 + S\theta_3 S\alpha_0 S\alpha_3)S\theta_1$$
$$= C\phi_2 C\psi_2$$

$$I_y = (C\theta_0 S\theta_3 + S\theta_0 C\theta_3 C\alpha_3)C\theta_1 + (-S\theta_0 S\theta_3 C\alpha_0 + C\theta_0 C\theta_3 C\alpha_0 C\alpha_3 - C\theta_3 S\alpha_0 S\alpha_3)S\theta_1$$
$$= C\phi_2 S\psi_2 S\Gamma_2 - S\phi_2 C\Gamma_2$$

$$I_z = S\theta_0 S\alpha_3 C\theta_1 + (C\theta_0 C\alpha_0 S\alpha_3 + S\alpha_0 C\alpha_3)S\theta_1 = C\phi_2 S\psi_2 C\Gamma_2 + S\phi_2 S\Gamma_2$$

$$J_x = -(C\theta_0 C\theta_3 - S\theta_0 S\theta_3 C\alpha_3)S\theta_1 + (-S\theta_0 C\theta_3 C\alpha_0 - C\theta_0 S\theta_3 C\alpha_0 C\alpha_3 + S\theta_3 S\alpha_0 S\alpha_3)C\theta_1$$
$$= S\phi_2 C\psi_2$$

$$J_y = -(C\theta_0 S\theta_3 + S\theta_0 C\theta_3 C\alpha_3)S\theta_1 + (-S\theta_0 S\theta_3 C\alpha_0 + C\theta_0 C\theta_3 C\alpha_0 C\alpha_3 - C\theta_3 S\alpha_0 S\alpha_3)C\theta_1$$
$$= S\phi_2 S\psi_2 S\Gamma_2 + C\phi_2 C\Gamma_2$$

$$J_z = -S\theta_0 S\alpha_3 S\theta_1 + (C\theta_0 C\alpha_0 S\alpha_3 + S\alpha_0 C\alpha_3)C\theta_1 = S\phi_2 S\psi_2 C\Gamma_2 - C\phi_2 S\Gamma_2$$

$$K_x = S\theta_0 C\theta_3 S\alpha_0 + C\theta_0 S\theta_3 S\alpha_0 C\alpha_3 + S\theta_3 C\alpha_0 S\alpha_3 = -S\psi_2$$

$$K_y = S\theta_0 S\theta_3 S\alpha_0 - C\theta_0 C\theta_3 S\alpha_0 C\alpha_3 - C\theta_3 C\alpha_0 S\alpha_3 = C\psi_2 S\Gamma_2$$

$$K_z = -C\theta_0 S\alpha_0 S\alpha_3 + C\alpha_0 C\alpha_3 = C\psi_2 C\Gamma_2$$

$$P_x = -a_2 = a_1 C\theta_1(C\theta_0 C\theta_3 - S\theta_0 S\theta_3 C\alpha_3) + a_1 S\theta_1(-S\theta_0 C\theta_3 C\alpha_0 - C\theta_0 S\theta_3 C\alpha_0 C\alpha_3$$
$$+ S\theta_3 S\alpha_0 S\alpha_3) + b_1(S\theta_0 C\theta_3 S\alpha_0 + C\theta_0 S\theta_3 S\alpha_0 C\alpha_3 + S\theta_3 C\alpha_0 S\alpha_3) + a_0 C\theta_0 C\theta_3$$
$$- a_0 S\theta_0 S\theta_3 C\alpha_3 + b_0 S\theta_3 S\alpha_3 + a_3 C\theta_3$$

$$P_y = 0 = a_1 C\theta_1(C\theta_0 S\theta_3 + S\theta_0 C\theta_3 C\alpha_3) + a_1 S\theta_1(-S\theta_0 S\theta_3 C\alpha_0 + C\theta_0 C\theta_3 C\alpha_0 C\alpha_3$$
$$- C\theta_3 S\alpha_0 S\alpha_3) + b_1(S\theta_0 S\theta_3 S\alpha_0 - C\theta_0 C\theta_3 S\alpha_0 C\alpha_3 - C\theta_3 C\alpha_0 S\alpha_3) + a_0 C\theta_0 S\theta_3$$
$$+ a_0 S\theta_0 C\theta_3 C\alpha_3 - b_0 C\theta_3 S\alpha_3 + a_3 S\theta_3$$

$$P_z = 0 = a_1 C\theta_1 S\theta_0 S\alpha_3 + a_1 S\theta_1(C\theta_0 C\alpha_0 S\alpha_3 + S\alpha_0 C\alpha_3) + b_1(-C\theta_0 S\alpha_0 S\alpha_3 + C\alpha_0 C\alpha_3)$$
$$+ a_0 S\theta_0 S\alpha_3 + b_0 C\alpha_3 + b_3$$

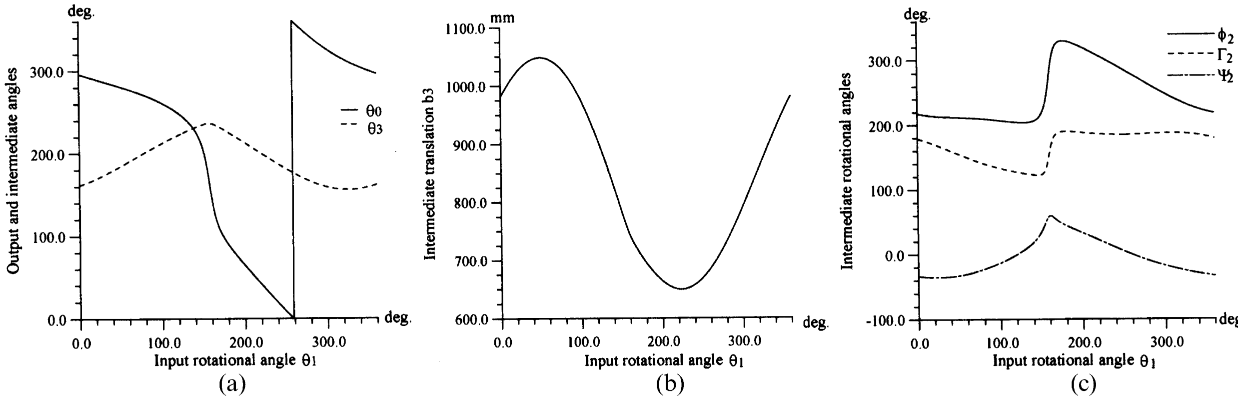

Fig. 6. Variation of intermediate displacements of RSCR mechanism.

To verify the proposed methodology, assume that the RSCR mechanism is characterized by the following parameters: $a_0=300$ mm, $a_1=35$ omm, $a_2=480$ mm, $a_3=250$ mm, $b_0=650$ mm, $b_1=150$ mm, $\alpha_0=35^{\circ}$ and $\alpha_3=-210^{\circ}$. The variations of the intermediate displacements in the RSCR mechanism are shown in Figs. 6(a), 6(b) and 6(c). The contour Σ_1 traced by the x_2 axis is presented in Fig. 7 in the form of a zenithal projection. The interference-free region bounded by Σ_1 can be obtained using the link shape-compensation method described in Section 2(C).

(B) RSCP mechanism

Figure 8 presents a schematic illustration of a RSCP mechanism. In accordance with the Kutzbach criterion [6], f=1. Table 2 summarizes the kinematic parameters of the various links within the mechanism. Comparing Tables 1 and 2, it can be inferred that the RSCR and RSCP mechanisms have the same governing equations. The only difference between the two mechanisms is that the output of the RSCR mechanism is a rotational angle (θ_0), whereas that of the RSCP mechanism is a translational motion (b_0). Therefore, the matrix given in Eq. (13) is applicable to both the RSCR mechanism and the RSCP mechanism.

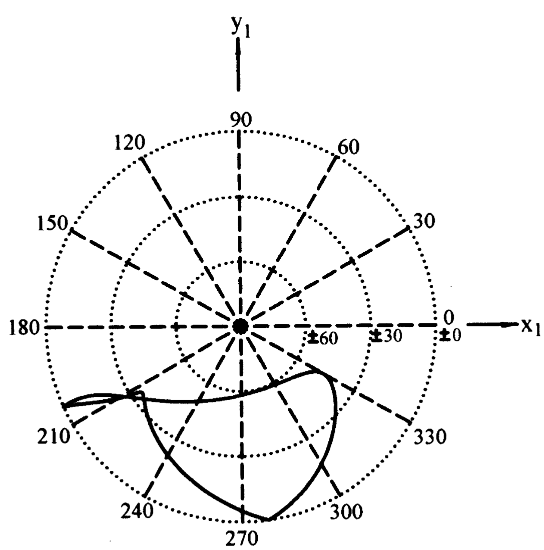

Fig. 7. Contour $\sum_1$ traced by x_2 axis of RSCR mechanism.

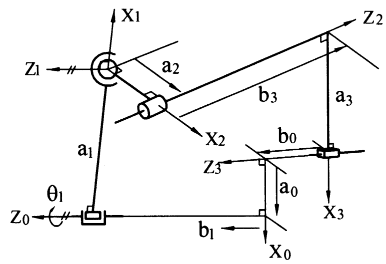

Fig. 8. Illustration of RSCP mechanism.

Table 2. Kinematic parameters of RSCP mechanism

Parameter	Link 1	Link 2	Link 3	Link 0
Link type	R-S	S-C	C-P	P-R
Distance b_i	b_1		b_3	b_0
Angle θ_i	θ_1		θ_3	θ_0
Length a_i	a_1	a_2	a_3	a_0
Twist angle α_i	0		α_3	α_0
Yaw Γ_i		Γ_2		
Pitch ψ_i		ψ_2		
Roll ϕ_i		ϕ_2		

(C) CSSP mechanism

Figure 9 illustrates a CSSP mechanism with f=3. The corresponding kinematic parameters are summarized in Table 3, where a_1, a_2, a_3, a_0, θ_0 and α_0 are link parameters, while b_0, θ_1, b_1, ϕ_2, ψ_2, Γ_2 , ϕ_3, ψ_3 and $\Gamma_3$are joint variables. Of these parameters, θ_1 and b_1 represent the inputs of the mechanism, while b_0 is the output. The governing equations for the CSSP mechanism can be obtained by post-multiplying both sides of Eq. (12) by $({}^2A_3{}^3A_0)^{-1}$ to obtain the following:

$$
{}^0\mathbf{A}_1{}^1\mathbf{A}_2=\begin{bmatrix} I_x^{\bullet} & J_x^{\bullet} & K_x^{\bullet} & P_x^{\bullet} \\ I_y^{\bullet} & J_y^{\bullet} & K_y^{\bullet} & P_y^{\bullet} \\ I_z^{\bullet} & J_z^{\bullet} & K_z^{\bullet} & P_z^{\bullet} \\ 0 & 0 & 0 & 1 \end{bmatrix}={}^0\mathbf{A}_3{}^3\mathbf{A}_2=\begin{bmatrix} I_x & J_x & K_x & P_x \\ I_y & J_y & K_y & P_y \\ I_z & J_z & K_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{14}
$$

where

$$
I_x=C\theta_0C\phi_3C\psi_3+S\theta_0(C\phi_3S\psi_3S\Gamma_3-S\phi_3C\Gamma_3)=I_x^{\bullet}=C\theta_1C\phi_2C\psi_2-S\theta_1S\phi_2C\psi_2
$$

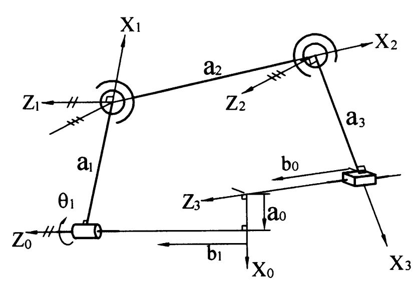

Fig. 9. Illustration of CSSP mechanism.

Table 3. Kinematic parameters of CSSP mechanism

Parameter	Link 1	Link 2	Link 3	Link 0
Link type	C-S	S-S	S-P	P-C
Distance b_i	b_1		0	b_0
Angle θ_i	θ_1		0	θ_0
Length a_i	a_1	a_2	a_3	a_0
Twist angle α_i	0			α_0
Yaw Γ_i		Γ_2	Γ_3	
Pitch ψ_i		ψ_2	ψ_3	
Roll ϕ_i		ϕ_2	ϕ_3	

$$I_y = -S\theta_0 C\alpha_0 C\phi_3 C\psi_3 + C\theta_0 C\alpha_0 (C\phi_3 S\psi_3 S\Gamma_3 - S\phi_3 C\Gamma_3) + S\alpha_0 (C\phi_3 S\psi_3 C\Gamma_3 + S\phi_3 S\Gamma_3)$$
$$= I_y^{\bullet} = S\theta_1 C\phi_2 C\psi_2 + C\theta_1 S\phi_2 C\psi_2$$

$$I_z = S\theta_0 S\alpha_0 C\phi_3 C\psi_3 - C\theta_0 S\alpha_0 (C\phi_3 S\psi_3 S\Gamma_3 - S\phi_3 C\Gamma_3) + C\alpha_0 (C\phi_3 S\psi_3 C\Gamma_3 + S\phi_3 S\Gamma_3)$$
$$= I_z^{\bullet} = -S\psi_2$$

$$J_x = C\theta_0 S\phi_3 C\psi_3 + S\theta_0 (S\phi_3 S\psi_3 S\Gamma_3 + C\phi_3 C\Gamma_3)$$
$$= J_x^{\bullet} = C\theta_1 (C\phi_2 S\psi_2 S\Gamma_2 - S\phi_2 C\Gamma_2) - S\theta_1 (S\phi_2 S\psi_2 S\Gamma_2 + C\phi_2 C\Gamma_2)$$

$$J_y = -S\theta_0 C\alpha_0 S\phi_3 C\psi_3 + C\theta_0 C\alpha_0 (S\phi_3 S\psi_3 S\Gamma_3 + C\phi_3 C\Gamma_3) + S\alpha_0 (S\phi_3 S\psi_3 C\Gamma_3 - C\phi_3 S\Gamma_3)$$
$$= J_y^{\bullet} = S\theta_1 (C\phi_2 S\psi_2 S\Gamma_2 - S\phi_2 C\Gamma_2) + C\theta_1 (S\phi_2 S\psi_2 S\Gamma_2 + C\phi_2 C\Gamma_2)$$

$$J_z = S\theta_0 S\alpha_0 S\phi_3 C\psi_3 - C\theta_0 S\alpha_0 (S\phi_3 S\psi_3 S\Gamma_3 + C\phi_3 C\Gamma_3) + C\alpha_0 (S\phi_3 S\psi_3 C\Gamma_3 - C\phi_3 S\Gamma_3)$$
$$= J_z^{\bullet} = C\psi_2 S\Gamma_2$$

$$K_x = -C\theta_0 S\psi_3 + S\theta_0 C\psi_3 S\Gamma_3$$
$$= K_x^{\bullet} = C\theta_1 (C\phi_2 S\psi_2 C\Gamma_2 + S\phi_2 S\Gamma_2) - S\theta_1 (S\phi_2 S\psi_2 C\Gamma_2 - C\phi_2 S\Gamma_2)$$

$$K_y = S\theta_0 C\alpha_0 S\psi_3 + C\theta_0 C\alpha_0 C\psi_3 S\Gamma_3 + S\alpha_0 C\psi_3 C\Gamma_3$$
$$= K_y^{\bullet} = S\theta_1 (C\phi_2 S\psi_2 C\Gamma_2 + S\phi_2 S\Gamma_2) + C\theta_1 (S\phi_2 S\psi_2 C\Gamma_2 - C\phi_2 S\Gamma_2)$$

$$K_z = -S\theta_0 C\alpha_0 S\psi_3 - C\theta_0 S\alpha_0 C\psi_3 S\Gamma_3 + C\alpha_0 C\psi_3 C\Gamma_3 = K_z^{\bullet} = C\psi_2 C\Gamma_2$$

$$P_x = -a_3 C\theta_0 - a_0 = P_x^{\bullet} = a_2 C\phi_2 C\psi_2 C\theta_1 - a_2 S\phi_2 C\psi_2 S\theta_1 + a_1 C\theta_1$$

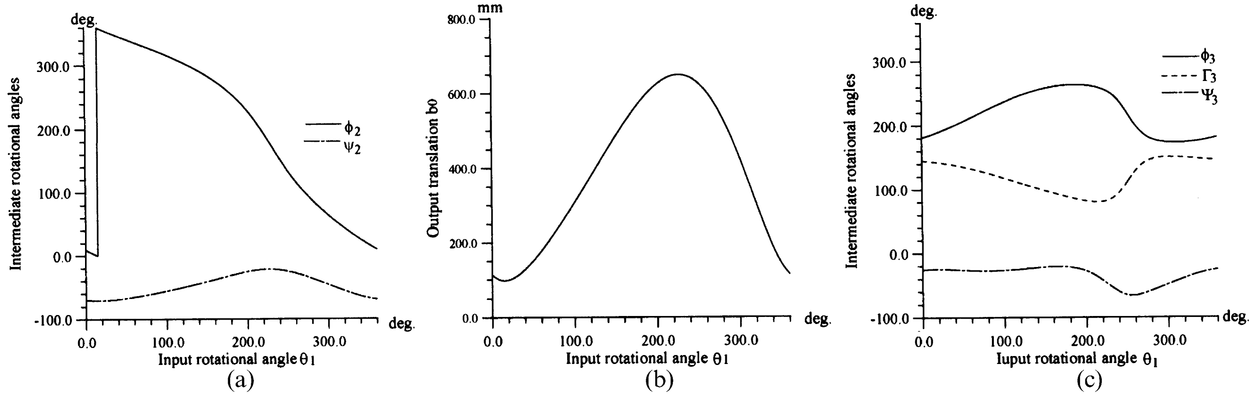

Fig. 10. Variation of intermediate displacements of CSSP mechanism.

$$P_y = a_3 S\theta_0 C\alpha_0 - b_0 S\alpha_0 = P_y^{\bullet} = a_2 C\phi_2 C\psi_2 S\theta_1 + a_2 S\phi_2 C\psi_2 C\theta_1 + a_1 S\theta_1$$

$$P_z = -a_3 S\theta_0 S\alpha_0 - b_0 C\alpha_0 = P_z^{\bullet} = -a_2 S\psi_2 + b_1.$$

By equating the two sides of Eq. (14), the governing equations for the positions and orientations of the links within the mechanism can be simultaneously obtained. Note that the spinning motion of link 2, i.e., Γ_2, does not affect the governing equations for the link positions since these equations are independent of Γ_2.

For illustration purposes, assume that the CSSP mechanism is characterized by the following parameters: $a_0 = 500$ mm, $a_1 = 300$ mm, $a_2 = 750$ mm, $a_3 = 350$ mm, $b_1 = 200$ mm, $\alpha_0 = 35^{\circ}$, $\theta_0 = -10^{\circ}$ and $\Gamma_2 = 0^0$. The corresponding variations of the intermediate displacements of the CSSP mechanism are shown in Figs. 10(a), 10(b) and 10(c). Meanwhile, the contour Σ_2 traced by thex_3 axis is shown in Fig. 11 in the form of a zenithal projection. Note that the jump of angle ϕ_2 in Fig. 10(a) from zero to 360° is due to the assumption that the CSSP mechanism rotates in the counter-clockwise direction (i.e., -1° is represented as 359°).

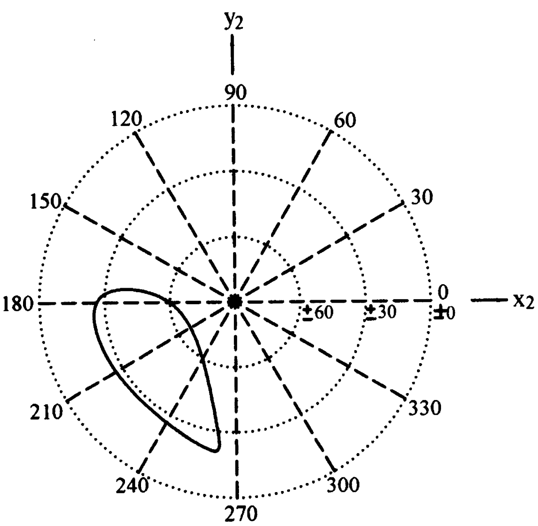

Fig. 11. Contour Σ_2 traced by x_3 axis of CSSP mechanism.

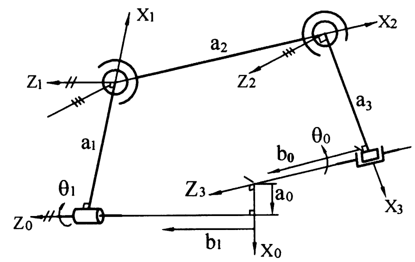

Fig. 12. Illustration of CSSR mechanism.

Table 4. Kinematic parameters of CSSR mechanism

Parameter	Link 1	Link 2	Link 3	Link 0
Link type	R-S	S-S	S-P	P-R
Distance b_i	b_1		0	b_0
Angle θ_i	θ_1		0	θ_0
Length a_i	a_1	a_2	a_3	a_0
Twist angle α_i	0			α_0
Yaw Γ_i		Γ_2	Γ_3	
Pitch ψ_i		ψ_2	ψ_3	
Roll ϕ_i		ϕ_2	ϕ_3	

(D) CSSR mechanism

Figure 12 illustrates a CSSR mechanism with f=3. Table 4 summarizes the kinematic parameters of the four links within the mechanism. Comparing Tables 3 and 4, it can be seen that the governing equations of the CSSP mechanism can also be applied to the CSSR mechanism. The only difference between the two mechanisms is that the output of the CSSR system is a rotational motion (θ_0), whereas that of the CSSP system is a translational motion (b_0).

4. CONCLUSION

This study has presented an extended D-H notation for modeling the kinematics of spatial mechanisms comprising prismatic, revolute, helical, cylindrical or spherical pairs. The validity of the proposed approach has been verified by modeling four generic kinematic chains, namely RSCR, RSCP, CSSR and CSSP, respectively. The results have shown that the extended D-H notation provides a viable basis for analytical or numerical schemes designed to examine the kinematic response of mechanical mechanisms.

REFERENCES

1. Denavit, J., and Hartenberg, R.S., "A kinematic notation for lower-pair mechanisms based on matrices," *Journal of Applied Mechanics*, Vol. 22, pp. 215–221, 1955.
2. Sheth, P.N. and Uicker, J.J., "A generalized symbolic notation for mechanisms," *Journal of Engineering for Industry, Transactions of the ASME*, Vol. 93, No. 1, pp. 102–112, 1971.
3. Sandor, G.N. and Erdman, A.G., "Advanced mechanism design: analysis and synthesis," Volume II, Prentice Hall, Inc., 1984.
4. Yang, A.T., "Displacement analysis of spatial five-link mechanisms using (3 × 3) matrices with dual-number element," *Journal of Engineering for Industry, Transactions of the ASME*, Vol. 91, No. 2, pp. 152–157, 1969.

5. Fischer, I.S., ‘‘Numerical analysis of displacements in spatial mechanisms with ball joints,’’ *Mechanism and Machine Theory*, Vol. 35, pp. 1623–1640, 2000.
6. Gupta, V., ‘‘Mobility analysis of a class of RPSPR kinematic chain,’’ *Journal of Mechanical Design, Transactions of the ASME*, Vol. 126, pp. 71–78, 2004.
7. Liu, C.H. and Cheng, S., ‘‘Direct singular positions of 3RPS parallel manipulators,’’ *Journal of Mechanical Design, Transactions of the ASME*, Vol. 126, pp. 1006–1016, 2004.
8. Akcali, I.D. and Multu, H., ‘‘A novel approach in the direct kinematics of stewart platform mechanisms with planar platforms,’’ *Journal of Mechanical Design, Transactions of the ASME*, Vol. 128, pp. 252–263, 2006.
9. Ting, K.-L. and Zhu, J., ‘‘On realization of spherical joints in RSSR mechanisms,’’ *Journal of Mechanical Design, Transactions of the ASME*, Vol. 127, pp. 924–930, 2005.
10. Paul, R.P., ‘‘Robot manipulators-mathematics, programming and control,’’ MIT press, Cambridge, Mass, 1982.
11. Hsieh, J.-F., ‘‘Application of homogenous transformation matrix to the design and machining of a geneva mechanism with curved slots,’’ *Proc. Instn. Mech. Engrs, Part C: J. Mechanical Engineering Science*, Vol. 221, pp. 1435–1443, 2007.
12. Waldron, K.J. and Kinzel, G.L., ‘‘Kinematics, dynamics and design of machinery, 2nd,’’ John Wiley & Sons, Inc., 2004.