# EXACT LINEARIZATION AND DISCRETIZATION OF NONLINEAR SYSTEMS SATISFYING A LAGRANGE PDE CONDITION

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## Abstract

A sufficient condition for exact linearization of a nonlinear system via an exponential transformation is obtained as a Lagrange partial differential equation. When its solution can be found, the transformation is determined such that the nonlinear systemis exactly converted into a linear system with arbitrary dynamics. When the transformation is invertible, this technique can be applied to exact discretization. Several examples are given to demonstrate the linearization and discretization processes and associated conditions. Asimulation result is presented to show that, under proper conditions, the obtained discrete-time model gives values that are identical to the continuous-time original at discrete-time instants for any sampling intervals.

**Keywords:** exact linearization; nonlinear systems; exponential variable transformation; exact discretization.

# LINÉARISATION ET DISCRÉTISATION EXACTE DE SYSTÈMES NON LINÉAIRES CONFORMES AUX CONDITIONS EDP DE LAGRANGE

## Résumé

Une condition suffisante pour une linéarisation exacte d'un système non linéaire via une transformation exponentielle est obtenue comme une équation partielle différentielle de Lagrange. Quand sa solution peut être obtenue, la transformation est déterminée de telle façon que le système non linéaire est exactement converti en un système linéaire avec une dynamique arbitraire. Quand la transformation est réversible, cette technique peut-être appliquée à une discrétisation exacte. Plusieurs exemples sont donnés pour démontrer le processus de linéarisation et de discrétisation et les conditions associées. Une simulation est présentée pour démontrer que, sous des conditions appropriées, le modèle en temps discret obtenu donne des valeurs qui sont identiques au modèle en temps continu, à des moments en temps discret pour tous intervalles d'échantillonnage.

**Mots-clés :** linéarisation exacte; systèmes non linéaires; transformation exponentielle variable; discrétisation exacte.

## 1 INTRODUCTION

Linearization of nonlinear systems has been one of the most important techniques used in the analysis and design of dynamic systems [1], since a large number of linear tools are readily available. This has also been the case in digital simulation and control of dynamic systems [2], where some form of discretization errors are often considered inevitable. However, linearization is usually carried out around a certain equilibrium state, at which no error is produced, while introducing approximation errors elsewhere. In contrast, exact linearization involves no such approximation errors anywhere. For instance, linearization via nonlinear feedback makes a nonlinear system act as a linear system under control [3]. Exact linearization via a variable transformation is another useful approach for some nonlinear systems [3]. For Riccati systems, Cole-Hopf transformation [4] and Nowakowski transformation [5] are used, although the first-order nonlinear system turns into a second-order linear system. These are special cases of bi-linear transformation, which is applicable to a larger class of nonlinear systems and can be viewed from the gauge invariance point of view [4]. A matrix version of the bi-linear transformation is used for exactly linearizing matrix differential Riccati equations, which appears when computing the optimal gain for state feedback control of linear plants [6]. In contrast to these transformations, fractional transformation used in [7] does not increase the order during linearization. Although this transformation is applicable to a more general Bernoulli equation [8], dynamics of the linearized system is not adjustable. Exponential transformation proposed in [9] for first-order nonlinear systems, is more versatile than fractional transformation and considered in the present study as a way to linearize higher-order nonlinear systems. Although not expected to be applicable to all nonlinear systems, it is nevertheless important to continue to expand the class of systems that can be linearized exactly and to investigate conditions on the system for achieving exact linearization via the exponential variable transformation.

An important application of exact linearization is an exact discretization. While general approximate discretization techniques abound for linear systems [6], there are fewer models for nonlinear systems [10]. Arecent addition to the discretization of nonlinear systems is one that preserves gauge invariance [4]. Unfortunately, gauge invariance alone does not preserve exactness in the discretization process. An exact methodis preferred, when possible, especially for nonlinear systems where inaccuracy of the discrete-time model often calls for complex analysis and design of controllers [11]. In the area of digital control, where the use of a zero-order-hold and an ideal sampler is assumed, a linear continuous-time system is known to be represented exactly by the so-called step-invariant model [12]. Exactness under discretization can be preserved, therefore, for nonlinear systems that can be linearized exactly.

The paper is organized as follows: In section 2, a sufficient condition is presented for an$n$-th order nonlinear state equation that can be transformed into a linear state equation. This condition is explained in some detail for first, second, and third order cases using examples to demonstrate the linearization process and some conditions involved.An exact discrete-time model is presented then in Section 3, where numerical simulations are carried out to show that, under proper conditions, the exact model gives states that match exactly those of the analog state equation at discrete-time instants for any discrete-time interval.

## 2 EXACT LINEARIZATION

The nonlinear system considered in the present study is assumed to be expressed by the following $n$-th order state equation:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x},t), \tag{1}$$

where the state vector $\mathbf{x}$ and the vector of generally nonlinear functions $\mathbf{f}(\mathbf{x},t)$ are denoted, respectively, by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{f}(\mathbf{x},t) = \begin{bmatrix} f_1(\mathbf{x},t) \\ \vdots \\ f_n(\mathbf{x},t) \end{bmatrix}, \tag{2}$$

with $f_i$ being continuously differentiable. This system is to be transformed into a linear system given by

$$\frac{d\mathbf{z}(t)}{dt} = \mathbf{Mz}(t), \tag{3}$$

where $\mathbf{M}$ is an arbitrary, constant, $n \times n$ matrix. This conversion is to be achieved using a variable transformation of the following form:

$$\mathbf{z} = \exp\{\mathbf{MV}(\mathbf{x})\}\mathbf{1}, \tag{4}$$

where $\mathbf{V}$ is a diagonal $n \times n$ matrix function to be determined and

$$\mathbf{1} = [1 \quad \cdots \quad 1]^T. \tag{5}$$

The initial condition $\mathbf{z}(0)$ should satisfy Eq. (4), given $\mathbf{x}(0)$. The linearization is possible if, for the system given by Eq. (1), a function $\mathbf{V}$ that satisfies the following condition can be found:

$$f_1(\mathbf{x},t)\frac{\partial \mathbf{V}(\mathbf{x},t)}{\partial x_1} + \cdots + f_n(\mathbf{x},\mathrm{t})\frac{\partial \mathbf{V}(\mathbf{x},t)}{\partial x_n} = \mathbf{I}_n. \tag{6}$$

This can be easily shown as follows: When $\mathbf{V}$ and $\frac{d\mathbf{V}}{dt}$ commute, such as when $\mathbf{V}$ is diagonalas assumed above, it can be shown that

$$\frac{d\exp\{\mathbf{MV}\}}{dt} = \mathbf{M}\frac{d\mathbf{V}}{dt}\exp\{\mathbf{MV}\} \quad \left(= \mathbf{M}\exp\{\mathbf{MV}\}\frac{d\mathbf{V}}{dt}\right). \tag{7}$$

Therefore, differentiation of the new variable $\mathbf{z}$ defined by Eq. (4) with respect to time gives

$$\begin{aligned} \frac{d\mathbf{z}}{dt} &= \mathbf{M}\frac{d\mathbf{V}}{dt}\exp(\mathbf{MV})\mathbf{1} \\ &= \mathbf{M}\left\{f_1(\mathbf{x})\frac{\partial \mathbf{V}(\mathbf{x})}{\partial x_1} + \cdots + f_n(\mathbf{x})\frac{\partial \mathbf{V}(\mathbf{x})}{\partial x_n}\right\}\mathbf{z}, \end{aligned} \tag{8}$$

which must be identical to Eq. (3), yielding condition (6). This condition is a linear first-order partial differential equation (PDE), often called Lagrange PDE. This equation is known to be relatively easy to solve and may be solved using any one of known techniques, such as the method of characteristic curves [13]. However, there seem to be no general systematic ways to find a solution. Therefore, when a solution is found to this PDE, the linearizing transformation can be found. Otherwise, no conclusion can be drawn on the linearizability. In the following, the process of exact linearization presented above is explained in some detail for first, second, and third ordercases. While the first-order case is rather general, higher-order cases require individual treatment and it becomes more difficult to explain the procedure as the order increases.

### 2.1 First-Order Case [9]

For the first-order system given by

$$\frac{dx}{dt}=f(x), \tag{9}$$

thelinearizability condition (6) is

$$f(x)\frac{dv(x)}{dx}=1, \tag{10}$$

from which one obtains, with $\varphi$ being constant,

$$v=\phi(x)=\int\frac{dx}{f(x)}+\varphi. \tag{11}$$

Therefore, any first-order system can be linearized exactly when the reciprocal of function $f$ is integrable. The corresponding variable transformation is given then by

$$z=\exp(mv(x))=\exp\left(m\left(\int\frac{dx}{f(x)}+\varphi\right)\right), \tag{12}$$

where $m$ is arbitrary. With this transformation, the nonlinear system is exactly linearized as

$$\frac{dz}{dt}=mz. \tag{13}$$

**Example 1:**

Let us consider the scalar Riccati equation given by

$$\dot{x}=f(x)=ax^2+bx+c, \tag{14}$$

where $b^2-4ac<0$ with $a$, $b$, and $c$ being constant. The case where $b^2-4ac\geq 0$ was considered in [9]. In the present case, the system has no real equilibrium and the state diverges from any initial state. The linearizing transformation given by Eq. (4) is found to be

$$z=\exp\left(m\left(\int\frac{dx}{ax^2+bx+c}+\varphi\right)\right)=\exp\left(m\left(\frac{2}{\sigma}\tan^{-1}\frac{2ax+b}{\sigma}+\varphi\right)\right), \tag{15}$$

where $\varphi$ is constant and $\sigma$ is defined as

$$\sigma=\sqrt{-b^2+4ac}. \tag{16}$$

## 2.2 Second-Order Case

The linearizability condition (6) for a time-invariant second-order system is the following first-order, two-variable, linear, partial differential equation (PDE):

$$f_1(x_1,x_2)\frac{\partial \mathbf{V}(x_1,x_2)}{\partial x_1}+f_2(x_1,x_2)\frac{\partial \mathbf{V}(x_1,x_2)}{\partial x_2}=\mathbf{I}_2. \tag{17}$$

This is a special class of Lagrange PDE and a number of powerful methods exist to find a solution. To demonstrate the process of finding a linearizing transformation, let us consider aclass of systems, for ease of explanation, given by

$$\begin{cases}\dot{x}_1=f_{11}(x_1)f_{12}(x_2)\\ \dot{x}_2=f_{21}(x_1)f_{22}(x_2)\end{cases}, \tag{18}$$

for which the exact-differential-form solution is possible. The characteristic Eq. (13) for Eq. (17) can be written in a compact form as

$$\frac{f_{21}(x_1)}{f_{11}(x_1)}dx_1=\frac{f_{12}(x_2)}{f_{22}(x_2)}dx_2=f_{21}(x_1)f_{12}(x_2)\frac{dV_{ii}}{1}, \tag{19}$$

from which one can obtain $V_{ii}$. The off-diagonal elements, $V_{ij}$ $(i\neq j)$, are zero since $\mathbf{V}$ is diagonal. The first equality in Eq. (19) gives

$$\frac{f_{21}(x_1)}{f_{11}(x_1)}dx_1-\frac{f_{12}(x_2)}{f_{22}(x_2)}dx_2=0, \tag{20}$$

which leads to the general solution given by

$$\xi(x_1,x_2)=\int\frac{f_{21}(x_1)}{f_{11}(x_1)}dx_1-\int\frac{f_{12}(x_2)}{f_{22}(x_2)}dx_2=c_1 \tag{21}$$

with $c_1$ being constant. When this equation can be solved, for instance, for $x_1$ as

$$x_1=\varsigma_1(x_2), \tag{22}$$

$V_{ii}$ can be found, using Eq. (19), as

$$V_{ii} = \int \frac{dx_2}{f_{21}(\varsigma_1(x_2))f_{22}(x_2)} + \varphi_{ii}(\xi(x_1,x_2)), \tag{23}$$

where $\varphi_{ii}(\xi(x_1,x_2))$ is an arbitrary function of its argument. Alternatively, if Eq. (21) can be solved for $x_2$ as $x_2 = \varsigma_2(x_1)$, then

$$V_{ii} = \int \frac{dx_1}{f_{11}(x_1)f_{12}(\varsigma_2(x_1))} + \varphi_{ii}(\xi(x_1,x_2)), \tag{24}$$

Since **V** is diagonal, its off-diagonal elements are zero.

**Example 2:**

Consider the nonlinear system given by

$$\begin{cases} \dot{x}_1 = ax_2 \\ \dot{x}_2 = \dfrac{bx_2^3}{(x_1+c)^2} \end{cases}, \tag{25}$$

where $x_1 \neq -c$. The equilibrium state is $x_2 = 0$ for all $x_1$ but $x_1 \neq -c$. The exact-form characteristic equations give

$$\xi(x_1,x_2) = \frac{1}{a}\int \frac{dx_1}{(x_1+c)^2} - \frac{1}{b}\int \frac{1}{x_2^2}dx_2 = -\frac{1}{a(x_1+c)} + \frac{1}{bx_2} = c_1, \tag{26}$$

where $c_1$ must be chosen to satisfy the initial condition; i.e., $c_1 = \xi(x_1(0),x_2(0))$. Solving the above equation for, say, $x_2$ and substituting it into Eq. (24), $V_{ii}$ can be foundas

$$\begin{aligned} V_{ii} &= \frac{1}{a}\int \frac{dx_1}{x_2} + \varphi_{ii}(\xi(x_1,x_2)) \\ &= \frac{b}{a^2}\left(\int \frac{1}{x_1+c}dx_1 + ac_1x_1\right) + \varphi_{ii}(\xi(x_1,x_2)) \\ &= \frac{b}{a^2}\ln(\kappa(x_1+c)) + \frac{bc_1}{a}x_1 + \varphi_{ii}(\xi(x_1,x_2)),. \end{aligned} \tag{27}$$

where

$$\kappa = \begin{cases} 1 & (x_1(0) > -c) \\ -1 & (x_1(0) < -c) \end{cases}. \tag{28}$$

Since the state trajectory cannot cross the line $x_1 = -c$, the sign of $\kappa$ depends only on the initial condition $x_1(0)$. When one chooses, for example, as

$$\mathbf{M} = -\mathbf{I}_2,\ \varphi_{11} = 0,\ \varphi_{22} = \xi(x_1, x_2), \tag{29}$$

a linearizing variable transformation is obtained as

$$\mathbf{z} = \exp\{\mathbf{MV}\}\mathbf{1} = \begin{bmatrix} \exp\left(-\dfrac{bc_1}{a}x_1\right)(\kappa(x_1+c))^{-\frac{b}{a^2}} \\ \exp\left(-\dfrac{1}{bx_2} + \dfrac{1}{a(x_1+c)}\right)\cdot\exp\left(-\dfrac{bc_1}{a}x_1\right)(\kappa(x_1+c))^{-\frac{b}{a^2}} \end{bmatrix}. \tag{30}$$

### 2.3 Third-Order Case

The linearizability condition (6) for a third-order nonlinear system isgiven by

$$f_1(x_1,x_2,x_3)\frac{\partial \mathbf{V}(x_1,x_2,x_3)}{\partial x_1} + f_2(x_1,x_2,x_3)\frac{\partial \mathbf{V}(x_1,x_2,x_3)}{\partial x_2} + f_3(x_1,x_2,x_3)\frac{\partial \mathbf{V}(x_1,x_2,x_3)}{\partial x_3} = \mathbf{I}_3. \tag{31}$$

As in the second-order case, let us consider a class of systems given inthe following form:

$$\begin{cases} \dot{x}_1 = f_1 = f_{11}(x_1)f_{12}(x_2)f_{13}(x_3) \\ \dot{x}_2 = f_2 = f_{21}(x_1)f_{22}(x_2)f_{23}(x_3) \\ \dot{x}_3 = f_3 = f_{31}(x_1)f_{32}(x_2)f_{33}(x_3) \end{cases}. \tag{32}$$

State variables are often chosen such that $\dot{x}_1 = x_2$ and $\dot{x}_2 = x_3$, so that the above system is special only in the form of $f_3$. The corresponding characteristic equation can be written as

$$\frac{dx_1}{f_{11}(x_1)f_{12}(x_2)f_{13}(x_3)} = \frac{dx_2}{f_{21}(x_1)f_{22}(x_2)f_{23}(x_3)} = \frac{dx_3}{f_{31}(x_1)f_{32}(x_2)f_{33}(x_3)} = \frac{dV_{ii}}{1}. \tag{33}$$

While this may be solved in different ways, it is convenient to use the fact that the above is also equal to

$$= \frac{g_1(x_1,x_2,x_3)dx_1 + g_2(x_1,x_2,x_3)dx_2 + g_3(x_1,x_2,x_3)dx_3}{g_1(x_1,x_2,x_3)f_1 + g_2(x_1,x_2,x_3)f_2 + g_3(x_1,x_2,x_3)f_3}, \tag{34}$$

for some functions $g_1$, $g_2$, and $g_3$ of states $x_1$, $x_2$ *and* $x_3$. The denominator of Eq. (34) can always be set to zero by proper choices of $g_i$, for which the numerator should be equated to zero so that itis a total derivative of some function. In this case, the solution to

$$g_1(x_1,x_2,x_3)dx_1 + g_2(x_1,x_2,x_3)dx_2 + g_3(x_1,x_2,x_3)dx_3 = 0 \tag{35}$$

can be found easily as $\xi_1(x_1,x_2,x_3) = c_1$. When thiscan be rewritten for one of the states, the third-order linearization problem reduces to that of the second-order. For instance, if $x_2$ can be expressed in terms of $x_1$and $x_3$, as

$$x_2 = \tau_1(x_1, x_3), \tag{36}$$

a solution to Eq. (33) can be rewritten as

$$\xi_2(x_1, x_3) = \int \frac{dx_1}{f_{11}(x_1) f_{12}(\tau_1(x_1, x_3)) f_{13}(x_3)} - \int \frac{dx_3}{f_{31}(x_1) f_{32}(\tau_1(x_1, x_3)) f_{33}(x_3)} = c_2. \tag{37}$$

When this can be solved for either $x_1$ or $x_3$, $V_{ii}$ can be determined. For example, if

$$x_1 = \tau_2(x_3), \tag{38}$$

it follows that

$$V_{ii} = \int \frac{dx_3}{f_{31}(\tau_2(x_3)) f_{32}(\tau_1(\tau_2(x_3), x_3)) f_{33}(x_3)} + \varphi_{ii}(\xi_1, \xi_2), \tag{39}$$

where $\varphi_{ii}$ is an arbitrary function of its arguments. The off-diagonal elements of $\mathbf{V}$ are zero.

**Example 3:**

Consider the nonlinear system given by

$$\begin{cases} \dot{x}_1 = f_1(x_2) & = x_2^{-1} \\ \dot{x}_2 = f_2(x_3) & = x_3 \\ \dot{x}_3 = f_3(x_1, x_3) & = a \exp(x_1)(x_3 + 1) \end{cases}. \tag{40}$$

The characteristic equation to the linearizability condition (31) is

$$\frac{dx_1}{x_2^{-1}} = \frac{dx_2}{x_3} = \frac{dx_3}{a \exp(x_1)(x_3 + 1)} = \frac{dV_{ii}}{1}, \tag{41}$$

with the additional relationship of

$$= \frac{(-a \exp(x_1) x_2) dx_1 + (-a \exp(x_1)) dx_2 + dx_3}{(-a \exp(x_1) x_2) x_2^{-1} + (-a \exp(x_1)) x_3 + a \exp(x_1)(x_3 + 1)}, \tag{42}$$

where the functions $g_i$ has been chosen to set the denominator to zero. By setting the numerator to zero in Eq. (42), one obtains

$$\xi_1(x_1, x_2, x_3) = -a x_2 \exp(x_1) + x_3 = c_1, \tag{43}$$

where the initial state condition must be satisfied; i.e., $c_1 = \xi_1(x_1(0), x_2(0), x_3(0))$. It should be noted that in the above process, $-a x_2 \int \exp(x_1) dx_1$ and $-a \exp(x_1) \int dx_2$ give the common term $-a x_2 \exp(x_1)$, while Eq. (43) can be solved for $x_2$ as

$$x_2 = \tau_1 = \frac{x_3 - c_1}{a} \exp(-x_1). \tag{44}$$

Thus, Eq. (37) as applied to Eq. (41) gives

$$\xi_2(x_1, x_3) = x_1 - \frac{1}{c_1+1} \ln \frac{|x_3 - c_1|}{|x_3+1|} = x_1 - \frac{1}{c_1+1} \ln\left(\kappa_1 \frac{x_3 - c_1}{x_3+1}\right) = c_2, \tag{45}$$

where $c_2(0) = \xi_2(x_1(0), x_3(0))$, and it must be assumed that $c_1 \neq -1$. In the above

$$\kappa_1 = \begin{cases} 1 & (x_3 \geq c_1 \text{ and } x_3 \geq -1, \text{ or } x_3 < c_1 \text{ and } x_3 < -1) \\ -1 & (x_3 < c_1 \text{ and } x_3 \geq -1, \text{ or } x_3 \geq c_1 \text{ and } x_3 < -1) \end{cases}. \tag{46}$$

Equation (45) can be arranged as

$$\exp(x_1) = \exp(c_2)\left\{\kappa_1 \frac{x_3 - c_1}{x_3+1}\right\}^{\frac{1}{c_1+1}}, \tag{47}$$

so that $V_{ii}$ can be found, finally, as

$$V_{ii} = \frac{1}{a}\exp(-c_2)\kappa_1^{\frac{-1}{c_1+1}} \int \left\{ (x_3 - c_1)^{\frac{-1}{c_1+1}} (x_3+1)^{\frac{-c_1}{c_1+1}} \right\} dx_3 + \varphi_{ii}(\xi_1, \xi_2), \tag{48}$$

where $\varphi_{ii}$ is an arbitrary function and $V_{ij} = 0$ $(i \neq j)$. Choosing $\mathbf{M}$ and $\varphi_{ii}$, for example, as

$$\mathbf{M} = -\mathbf{I}_2,\ \varphi_{11} = 0,\ \varphi_{22} = \xi_2(x_1, x_3),\ \varphi_{33} = \xi_1(x_1, x_2, x_3), \tag{49}$$

a linearizing variable transformation is obtained as

$$\begin{cases} z_1 = & \exp\left\{ -\frac{1}{a}\exp(-c_2)\kappa_1^{\frac{-1}{c_1+1}} \int \left\{ (x_3 - c_1)^{\frac{-1}{c_1+1}} (x_3+1)^{\frac{-c_1}{c_1+1}} \right\} dx_3 \right\} \\ z_2 = & \exp(-x_1)\left(\kappa_1 \frac{x_3 - c_1}{x_3+1}\right)^{\frac{1}{c_1+1}} \cdot z_1 \\ z_3 = & \exp(a x_2 \exp(x_1) - x_3) \cdot z_1 \end{cases} \tag{50}$$

To actually compute the transformation, an initial condition $\mathbf{x}(0)$ must be specified. For example, if the initial states are such that $c_1 = c_2 = 0$, the transformation is given by

$$\begin{cases} z_1 = (\kappa_2 x_3)^{\frac{-1}{a\kappa_1}} \\ z_2 = \kappa_1 x_3 (x_3+1)^{-1} \exp(-x_1) \cdot z_1 \\ z_3 = \exp(a \exp(x_1) x_2 - x_3) \cdot z_1 \end{cases}. \tag{51}$$

where Eq. (46) becomes

$$\kappa_1 = \begin{cases} 1 & (x_3 < -1 \text{ or } x_3 \geq 0) \\ -1 & (-1 \leq x_3 < 0) \end{cases}. \tag{52}$$

and where

$$\kappa_2 = \begin{cases} 1 & (x_3 \geq 0) \\ -1 & (x_3 < 0) \end{cases}. \tag{53}$$

Thus, in this example, state $x_3$ must be monitored and the transformation changed based on this information.

## 3 EXACT DISCRETIZATION

Once a nonlinear system is linearized exactly, it can be discretized exactly using the so called step-invariant model. This exact discrete-time model plays a very important role, and is used commonly, in the design of a digital control system to replace an analog system, in both open-loop [12] and closed-loop settings [14].

### 3.1 Linear Case

When a linear continuous-time system is controlled using digital controllers, its input is held by a zero-order-hold (zoh) and output sampled by an ideal sampler. The combination of a zoh, a plant, and a sampler represents the so-called step-invariant discrete-time model, whose response matches exactly those of the continuous-time plant at discrete-time instants for any sampling interval [12]. For the linearized system given by Eq. (3), this exact discrete-time model is given by

$$\delta z_k = \frac{\exp(\mathbf{M}T) - \mathbf{I}}{T} z_k, \tag{54}$$

where $z_k$ is a discrete-time state vector with $k$ being an integer variable representing time steps, $\delta$ is the delta operator defined by

$$\delta = \frac{q-1}{T}, \tag{55}$$

$q$ the ordinary shift operator such that $qz_k = z_{k+1}$, and $T$ the sampling interval. The delta operator is used here, since it proves to be very convenient when relating discrete-time results to continuous-time results [6], [12]. The discrete-time integral gain, $(\exp(\mathbf{M}T) - \mathbf{I})/T$, in Eq. (54) is the average value of the state transition matrix during one sampling period and approaches an identity matrix as the sampling interval approaches zero.

### 3.2 First-Order Case [9]

The exact discrete-time model of a first order system given by Eq. (9) can be obtained by substituting the inverse relationship, $x_k = \phi^{-1}(v_k)$, of Eq. (11) to the exact linear modelof Eq. (13), as

$$\delta x_k = \frac{\phi^{-1}(\phi(x_k)+T)-x_k}{T}. \tag{56}$$

It must be emphasized that, in the discretization problem, the existence of the inverse transformation of exact linearization must be assumed.

### 3.3 Second and Higher-Order Cases

The exact discrete-time model for the second and higher-order cases can be obtained in a manner similar to that for the first-order case, except that a solution to simultaneous equations is required in addition to the inverse transformation. The resulting model can be written as

$$\delta x_k = \frac{1}{T}\{\psi(\phi(x_k)+T)-x_k\}, \tag{57}$$

Where $\Psi$ represents the vector of solutions to the simultaneous equations. It is often simpler to use the step-invariant model, $\delta\mathbf{V}_k=\mathbf{I}$, of the linear model $d\mathbf{V}/dt=\mathbf{I}$, than use Eq. (57), as illustrated below:

**Example 4:**

To show the basic concept, consider the third-order system used in Example 3, where $a=-0.2$, $x_1(0)=-\ln 2$, $x_2(0)=-10$, and $x_3(0)=1$. The initial states of the nonlinear system are so chosen for convenience that $c_1=c_2=0$ and $\kappa_1=\kappa_2=1$. In this case Eq. (51) can be inverted as

$$\begin{cases} x_1 = V_{22} - \ln\dfrac{\exp(aV_{11})+1}{\exp((a-1)V_{11})} \\ x_2 = \dfrac{(\exp(aV_{11})+1)(\exp(aV_{11})+V_{11}-V_{33})}{a\exp(V_{22})\exp((a-1)V_{11})} \\ x_3 = \exp(aV_{11}) \end{cases}. \tag{58}$$

The exact discrete-time model can be foundusing Eqs. (57) and (58). Asimpler way for the present case is to use the step-invariant discrete-time model $\delta\mathbf{V}_k=\mathbf{I}$, which gives

$$\begin{cases} \delta x_{1,k} = 1 + \dfrac{1}{T}\ln\dfrac{\exp((a-1)T)(x_{3,k}+1)}{\exp(aT)x_{3,k}+1} \\ \delta x_{2,k} = \dfrac{\exp(aT)-1}{aT}\dfrac{\left(\exp(aT)x_{3,k}^2+x_{3,k}-ax_{2,k}\exp(x_{1,k})\right)}{\exp(aT)\exp(x_{1,k})(x_{3,k}+1)} \\ \delta x_{3,k} = \dfrac{\exp(aT)-1}{T}x_{3,k} \end{cases}. \tag{59}$$

It can be shown, using relationships among the state variables, that the above exact discrete-time model can be rewritten as

$$\begin{cases} \delta x_{1,k} = \Gamma_1(x_{2,k},T)f_1(x_{2,k}) \\ \delta x_{2,k} = \Gamma_2(T)f_2(x_{3,k}) \\ \delta x_{3,k} = \Gamma_3(T)f_3(x_{1,k},x_{3,k}) \end{cases}, \tag{60}$$

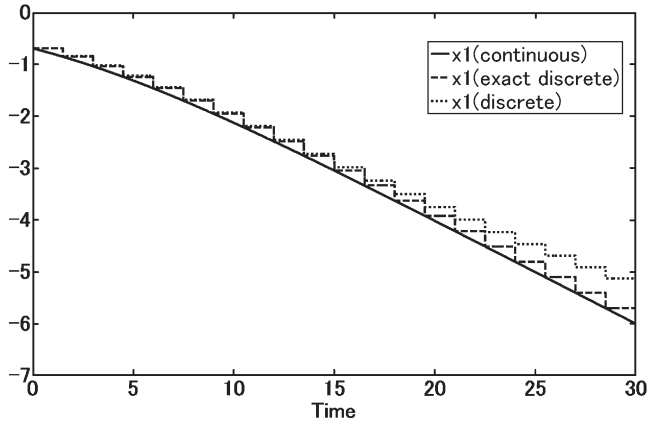


Fig. 1. The response of state $x_1$; continuous-time, exact discrete-time, and forward-difference, models.

where

$$\begin{cases} \Gamma_1(x_{2,k},T) = -\dfrac{x_{2,k}}{T}\ln\left(1 - T\dfrac{\exp(-aT)\Gamma_2}{x_{2,k}}\right) & \to 1\ (T\to 0) \\ \Gamma_2(T) = \Gamma_3(T) = \dfrac{\exp(aT)-1}{aT} & \to 1\ (T\to 0) \end{cases}. \tag{61}$$

Similarity of expression between the discrete-time form, Eq. (60), and the continuous-time form, Eq. (40) is apparent in view of Eq. (61).

Simulation results are given in Figs. 1–3 , where the solid lines (denoted as "continuous") show the state responses of the continuous-time system, while the dotted and dashed lines are those of the discrete-time models sampled and held at $T = 1.5$ s The dotted lines ("discrete") are for the forward-difference model, where the derivative is simply replaced with its Euler approximate; that is, $\Gamma_i$s are fixed at unities in Eq. (60). Although applicable to most nonlinear systems, the forward-difference model has errors that tend to grow as the sampling interval increases. The dashed lines are for the proposed exact model, whose responses match exactly

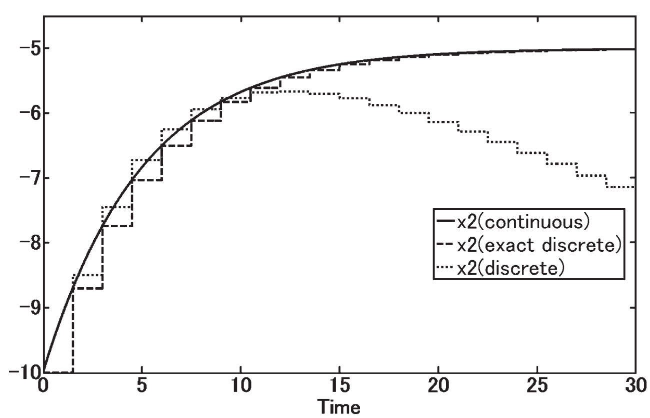


Fig. 2. The response of state $x_2$; continuous-time, exact discrete-time, and forward-difference, models.

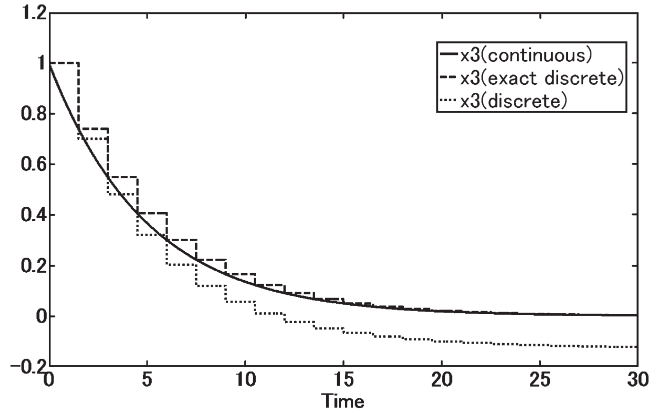


Fig. 3. The response of state $x_3$; continuous-time, exact discrete-time, and forward-difference, models.

those of the continuous-time original at discrete instants. By the way, the simulation results using the model given by Eq. (59) are exactly the same as those using Eq. (60).

## 4 CONCLUSIONS

A sufficient condition has been presented for exactly converting a nonlinear systeminto a linear system with arbitrary dynamicsthroughan exponential variable transformation. Thistransformation can be determinedwhen a solution is found to aLagrange partial differential equation, for which a number of methods exist. This techniqueand associated conditions has been demonstrated using first, second, and third-order examples in some detail. As an important application of this linearization, exact discretization of nonlinear systems has beenpresented, where the inverse transformation must exist. In general, the discrete-time model needs to be switched by monitoring its states. A numerical example, where this switching is not required, has been shown, where the state responses of the exact discrete-time systemgave exact values at the discrete-time instants for any discrete-time interval.

Not all nonlinear systems are expected to be linearizableor discretizable exactly, but the work presented here illustrates a way to enlarge a class of systems for which these can be achieved. Furthermore, the concept of variable transformation is not limited to nonlinear systems. Since a linear system can be transformed into another linear system with arbitrary dynamics, this may be considered as a problem of system modeling in combination with some sort of control. Moreover, from an exact discretization point of view, there is no need to convert a nonlinear system into a linear system, but it suffices to transformed into another nonlinear system whose exact discrete-time model is known. These aspects need further investigation.

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