DIRECT KINEMATIC ANALYSIS OF A FAMILY OF 4-DOF PARALLEL MANIPULATORS WITH A PASSIVE CONSTRAINING LEG

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Received February 2011, Accepted October 2011
No. 11-CSME-17, E.I.C. Accession 3257

ABSTRACT

This paper presents direct kinematic analysis of a family of 3R1T parallel manipulators, while R and T denote the rotational and translational degrees of freedom respectively. The manipulators consist of two rigid bodies, a movable platform and a fixed (base) connected to each other by four active legs and one constraining passive leg. First, the direct position kinematics of the manipulators is analyzed. For a general manipulator of this class, this analysis results in a univariate polynomial of degree 30 along with a set of other univariate polynomials of degree 16 and 4 respectively. However, for a special architecture of the manipulators, it is shown that the direct position kinematics leads to a minimal univariate polynomial of degree 12. A numerical example is also included to confirm the results. Moreover, direct velocity and direct kinematic singularities of the manipulators are analyzed using Jacobian matrices.

Keywords: parallel manipulators; direct kinematics; Jacobian matrices; velocity analysis; singularity analysis.

ANALYSE CINÉMATIQUE DIRECTE D'UNE FAMILLE DE MANIPULATEURS PARALLÈLES À 4-DDL AVEC UN MEMBRE PASSIF CONTRAIGNANT

RÉSUMÉ

Cet article présente une analyse cinématique directe d'une famille de manipulateurs parallèles, 3R1T, R et T étant respectivement de degré de liberté rotationnel et translationnel. Le manipulateur consiste en deux corps rigides, d'une plateforme mobile et d'une base fixe connectés par quatre membres actifs et un membre passif contraignant. Pour commencer, la cinématique directe des manipulateurs est analysée. Pour un manipulateur général de cette classe, le résultat de l'analyse est un polynôme en une seule variable de degré 30, avec un ensemble d'autres polynômes en une seule variable, de degrés 16 et 4. Toutefois, pour une architecture spéciale des manipulateurs, il est démontré que la cinématique directe conduit à un polynôme minimal en une variable, de degré 12. Un exemple numérique est aussi inclus pour confirmer les résultats. De plus, les singularités des manipulateurs sont analysées à l'aide des matrices jacobiennes.

Mots-clés : Manipulateurs parallèles; cinématique directe; matrices jacobiennes; analyse de vitesses; analyse des singularités.

1. INTRODUCTION

A parallel manipulator is a mechanism composed of a moving platform connected to a fixed one by means of at least two limbs. Parallel manipulators have received more and more attention over the last two decades. This popularity is a result of the fact that the parallel manipulators have more advantages in comparison to serial manipulators in many aspects, such as stiffness in mechanical structure, high position accuracy, and high load carrying capacity. However, they have some drawbacks such as limited workspace and complex forward position kinematics problems. The most studied type of parallel manipulator is without doubt the so-called general Gough–Stewart platform, a fully parallel manipulator introduced by Gough as a universal tire-testing machine [1] and proposed as a flight simulator by Stewart [2]. On the other hand, it must be noted that in many industrial applications, such as some assembly operations, parallel manipulators with fewer degrees of freedom than six can be successfully used instead of the general Gough–Stewart platform, such as the famous DELTA and the Orthoglide parallel robots with pure translational motions [3–5], 4-DOF parallel manipulators with parallel active limbs [6–8, 12–15] and also spherical manipulators with pure rotational motions [9, 10].

One major area of application of parallel manipulators is flight and motion simulation. For this type of application, rotational freedoms play a major role, while translations are of lesser importance [11]. However, one translational freedom, the heave, is of great significance in flight simulation [11]. Hence, some researchers proposed a subset of platform freedoms for the purposes of flight simulation, namely, manipulators with three rotational and only one translation DOFs (3R1T parallel manipulators); see for instance [12–15].

The purpose of this paper is to study the direct kinematics of a family of 3R1T parallel manipulators having four active legs and one passive leg.

2. TOPOLOGY GENERATION

Many different approaches have been proposed for topology generation of parallel kinematic structures, such as methods based on displacement group theory [16], methods based on screw theory [17–18], vector approach [19] and the approach that is based on the addition of a passive leg [20].

The method of addition of a passive leg, which is applied in this paper, considers that the moving platform motion is constrained by a passive leg connected to it. In fact, the passive leg is carefully chosen in such a way that the number of degrees of freedom and type of available motions for the end-effector correspond to the desired ones.

In our case, we selected a passive leg composed of two joints: a prismatic (P) joint and a spherical (S) one. As a consequence, the moving platform has four degrees of freedom and is constrained to perform three rotations and one translation. If we apply four active legs and if it is assumed that each active leg has only two links and three joints then sum of degrees of freedom of the three joints is equal to six. By employing different joints in each leg, we can use prismatic (P), revolute (R), universal (U) and spherical (S) joints. Then, a family of parallel manipulators (Fig. 1) is formed by the following architectures: 4UPS+PS, 4PUS+PS, 4URS+PS and 4RUS+PS. An underlined letter is an active joint which states the presence of an actuator. One can observe that all actuators are at or near the base. To decrease the reaction forces in the passive prismatic joint, this joint can be replaced by a cylindrical one. Moreover, for most kinematicians, substitution of universal joints for spherical ones is a common practice. However from a technical point of view, this practice can produce undesirable situations. For example,

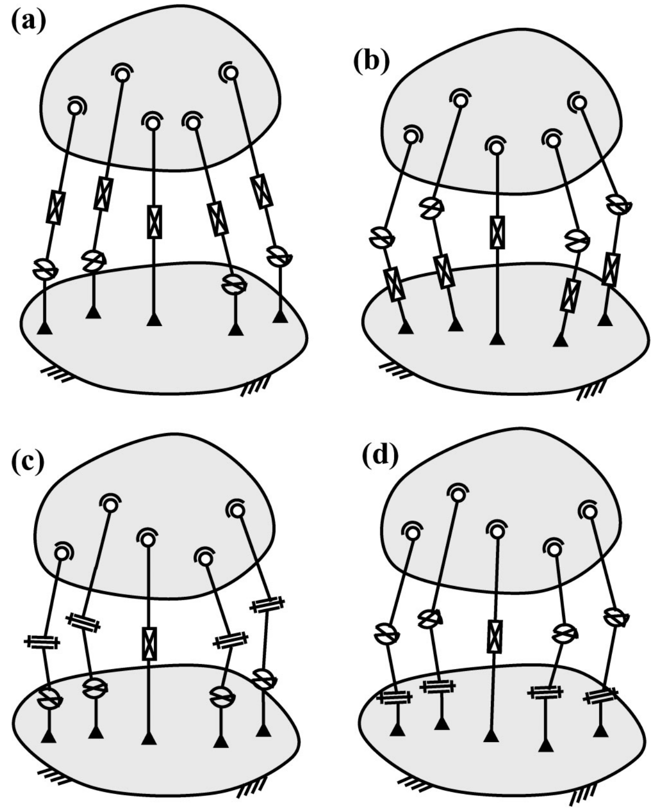

Fig. 1. Kinematic diagram of members of the family: (a) 4UPS+PS, (b) 4PUS+PS, (c) 4URS+PS (d) 4RUS+PS.

the two links connected by the prismatic joint in a SPS-type leg could turn uncontrollably about the rotation axis located between the two spherical joints.

3. DIRECT POSITION ANALYSIS

For parallel manipulators, the direct position analysis is stated as follows: a set of active joint variables is given and it is required to find the pose (position and orientation) of the moving platform. When the joint variables are assigned, the manipulators under study become the 4US+1PS structure. Accordingly, solution of direct position kinematics of the manipulators is equivalent to determination of assembly modes of the 4US+1PS structure.

The geometry of 4US+1PS structure, schematically shown in Fig. 2, is given. In particular, the length l_i of the line segment A_iB_i are known (for i=1,…, 4) where A_i and B_i denote the center of universal and spherical joints respectively. Position of points A_i (i=1,…, 4) are given in an arbitrary Cartesian reference coordinate frame $O\{x,y,z\}$ fixed to the base at point O while the z-axis is along the direction of prismatic joint of the passive leg. Without loss of generality, an arbitrary coordinate frame $P\{u,v,w\}$ is attached at the center of spherical joint of the passive leg denoted by P, while the u- and v- axes are located on the plane passing through points B_1, B_2 and P.

Since the moving platform of manipulators has no degrees of freedom in x and y direction, its position can be determined by vector $\mathbf{h}=[0 \quad 0 \quad h]^T$. In addition, by defining three unit vectors **e, f** and **g** as

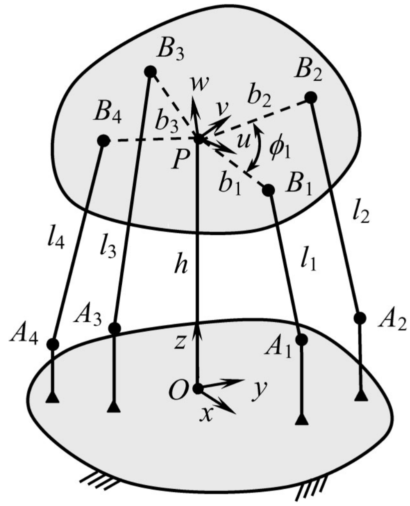

Fig. 2. The 4US+1PS structure.

$$\mathbf{e} = [e_1 \quad e_2 \quad e_3]^T = \frac{\overrightarrow{PB_1}}{|\overrightarrow{PB_1}|} \tag{1a}$$

$$\mathbf{f} = [f_1 \quad f_2 \quad f_3]^T = \frac{\overrightarrow{PB_2}}{|\overrightarrow{PB_2}|} \tag{1b}$$

$$\mathbf{g} = [g_1 \quad g_2 \quad g_3]^T = \frac{\overrightarrow{PB_3}}{|\overrightarrow{PB_3}|} \tag{1c}$$

the orientation of moving platform can be obtained by a unit vector **r** as

$$\mathbf{r} = [r_1 \quad r_2 \quad r_3]^T = \frac{\mathbf{e} \times \mathbf{f}}{|\mathbf{e} \times \mathbf{f}|} \tag{2}$$

while **r** is a unite vector along the *w*-axis and the sign "$\times$" denotes the cross product between vectors. So the pose of moving platform (and the closure of the 4US+1PS structure) can be uniquely parameterized by seven parameters h, e_i and f_i (i=1, 2, 3).

The closure equations for the structure are

$$\mathbf{a}_i + l_i \mathbf{l}_i = \mathbf{h} + \mathbf{b}_i, \mathrm{i} = 1, \ldots 4, \tag{3}$$

in which $\mathbf{l}_i$ is a unit vector representing the direction of vector $\overrightarrow{A_iB_i}$. Moreover, $\mathbf{a}_i$ is the position vector of the point A_i in the reference coordinate frame and is denoted as

$$\mathbf{a}_i = \overrightarrow{OA_i} = [\,a_{i1} \quad a_{i2} \quad a_{i3}\,]^T, \mathrm{i} = 1, \ldots 4, \tag{4}$$

Taking into account Eqs. (1), vectors $\mathbf{b}_i$ can be written accordingly as

$$\mathbf{b}_1 = b_1\mathbf{e} \tag{5a}$$

$$\mathbf{b}_2 = b_2\mathbf{f} \tag{5b}$$

$$\mathbf{b}_3 = b_3\mathbf{g} \tag{5c}$$

where $b_i = \overline{PB_i}$ (for i=1, 2, 3). Moreover, vector $\mathbf{b}_4$ can be defined as the linear combination of three other vectors $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$, as follows

$$\mathbf{b}_4 = z_1\mathbf{b}_1 + z_2\mathbf{b}_2 + z_3\mathbf{b}_3 \tag{6}$$

in which z_i (i=1, 2, 3) are three constant coefficients. Rearranging Eq. (3) yields

$$\mathbf{a}_i - \mathbf{h} - \mathbf{b}_i = -l_i\mathbf{l}_i, \mathrm{i} = 1, \ldots 4, \tag{7}$$

Please note that the vectors **e**, **f** and **g** are unit vectors; thus

$$\sum_{i=1}^{3} e_i^2 = 1 \tag{8a}$$

$$\sum_{i=1}^{3} f_i^2 = 1 \tag{8b}$$

$$\sum_{i=1}^{3} g_i^2 = 1 \tag{8c}$$

and

$$\sum_{i=1}^{3} e_i f_i = \delta_1 \tag{9a}$$

$$\sum_{i=1}^{3} e_i g_i = \delta_2 \tag{9b}$$

$$\sum_{i=1}^{3} f_i g_i = \delta_3 \tag{9c}$$

where $\delta_i = \cos\phi_i$ and ϕ_i is the angle between the corresponding unit vectors; for instance, ϕ_1 is the angle between vectors **e** and **f**, as shown in Fig. 2. Squaring two sides of Eqs. (7) and introducing Eqs. (1), (4)–(6), (8) and (9) into the resultant equations yields a system of four equations, namely,

$$s_1 e_1 + s_2 e_2 + s_3 e_3 + s_4 = 0 \tag{10}$$

$$s_5 f_1 + s_6 f_2 + s_7 f_3 + s_8 = 0 \tag{11}$$

$$s_9 g_1 + s_{10} g_2 + s_{11} g_3 + s_{12} = 0 \tag{12}$$

$$s_{13} e_1 + s_{14} e_2 + s_{15} e_3 + s_{16} f_1 + s_{17} f_2 + s_{18} f_3 + s_{19} g_1 + s_{20} g_2 + s_{21} g_3 + s_{22} = 0 \tag{13}$$

while the coefficients s_i depend on h and kinematic parameters of the manipulators and are presented in appendix A. Eqs. (10), (11) and (13) constitute a system of 3 linear equations with respect to parameters e_1, e_2 and f_1 which can be solved readily as

$$e_1 = \frac{t_1 e_3 + t_2 f_2 + t_3 f_3 + t_4}{t_5} \tag{14a}$$

$$e_2 = \frac{t_6 e_3 + t_7 f_2 + t_8 f_3 + t_9}{t_5} \tag{14b}$$

$$f_1 = -\frac{s_6 f_2 + s_7 f_3 + s_8}{s_5} \tag{14c}$$

and coefficients t_i (i=1,…,9) are presented in appendix B. Moreover, Eq. (12) can be solved for g_1 as

$$g_1 = -\frac{s_{10}g_2 + s_{11}g_3 + s_{12}}{s_9} \tag{15}$$

Now, introducing the above values of parameters e_1, e_2, f_1 and g_1 into Eqs. (8) and (9) leads to

$$u_1e_3^2 + u_2f_2^2 + u_3f_3^2 + u_4e_3f_2 + u_5e_3f_3 + u_6f_2f_3 + u_7e_3 + u_8f_2 + u_9f_3 + u_{10} = 0 \tag{16a}$$

$$u_{11}f_2^2 + u_{12}f_3^2 + u_{13}f_2f_3 + u_{14}f_2 + u_{15}f_3 + u_{16} = 0 \tag{16b}$$

$$u_{17}f_2^2 + u_{18}f_3^2 + u_{19}e_3f_2 + u_{20}e_3f_3 + u_{21}f_2f_3 + u_{22}e_3 + u_{23}f_2 + u_{24}f_3 + u_{25} = 0 \tag{16c}$$

$$u_{26}e_3 + u_{27}f_2 + u_{28}f_3 + u_{29} = 0 \tag{16d}$$

$$u_{30}f_2 + u_{31}f_3 + u_{32} = 0 \tag{16e}$$

and

$$G_1g_2^2 + G_2g_2 + G_3 = 0 \tag{17}$$

where coefficients u_i and G_i are presented in appendix C. Considering the values of coefficients s_i, t_i, u_i and G_i, one can see that Eqs. (16) (17) are, in fact, a system of six nonlinear equations in six parameters h, e_3, f_2, f_3, g_2 and g_3.

Variables f_2 and f_3 can be obtained from Eqs. (16d) and (16e), in terms of e_3, as follows

$$f_2 = \frac{v_1e_3 + v_2}{v_3} \tag{18}$$

$$f_3 = \frac{v_4e_3 + v_5}{v_3} \tag{19}$$

while coefficients v_i (i=1,…, 5) are listed in appendix D. Substituting the values of parameters f_2 and f_3 from Eqs. (18) and (19) into Eqs. (16a) – (16c) and rearranging the resultant equations yields a system of three nonlinear equations as

$$w_1e_3^2 + w_2e_3 + w_3 = 0 \tag{20}$$

$$w_4e_3^2 + w_5e_3 + w_6 = 0 \tag{21}$$

$$w_7e_3^2+w_8e_3+w_9=0 \tag{22}$$

and coefficients w_i (i=1,…, 9) are presented in appendix D. Taking $(20)\times w_4-(21)\times w_1$ and $(20)\times w_7-(22)\times w_1$ respectively and rearranging the resultant equations leads to

$$m_1e_3+m_2=0 \tag{23}$$

$$m_3e_3+m_4=0 \tag{24}$$

where coefficients m_i (i=1,…, 4) are also listed in appendix D. Obtaining the parameter e_3 from Eqs. (23) results in

$$e_3=-\frac{m_2}{m_1} \tag{25}$$

Taking into account the values of parameters w_i and m_i, if Eq. (25) is introduced into Eqs. (20) and (24) then the following system of equations is obtained

$$H_1g_2^4+H_2g_2^3+H_3g_2^2+H_4g_2+H_5=0 \tag{26}$$

$$H_6g_2^2+H_7g_2+H_8=0 \tag{27}$$

where coefficients H_i (i=1,…, 8) are polynomials of at most degree four in g_3. Now, Eqs. (17), (26) and (27) constitute a system of three nonlinear equations with respect to parameters g_2, g_3 and h.

Parameter g_2 can be eliminated from Eqs. (17) and (26) and form Eqs. (17) and (27) using Sylvester dialytic elimination method [21], see appendices E and F. The resultant equations are respectively:

$$\sum_{i=0}^{16} M_ig_3^i=0 \tag{28}$$

and

$$\sum_{i=0}^{4} N_ig_3^i=0 \tag{29}$$

where coefficients M_i and N_i are polynomials of at most degree four in h. Finally, variable g_3 can be eliminated from Eqs.(28) and (29) again by Sylvester dialytic elimination method. This leads to a thirty -order polynomial in the unknown h as follows

$$\sum_{i=0}^{30} R_i h^i = 0 \tag{30}$$

where R_i depends only on kinematic parameters of the manipulators. The detailed expressions for R_i are not given here because they are too large to serve any useful purpose. What is important to point out here is that the above equation admits at most thirty six solutions for h while some of them may be complex.

Once h is found, the value(s) of g_3 can be calculated from Eqs. (28) and (29) by setting the greatest common divisor of these equations to be zero, In addition, the unique value of the other variables of vectors **e**, **f**, **g** and **r** can be calculated from Eqs. (1), (25), (18), (19), (14) and (15). Please note that only the solutions are admissible for which Eqs. (8) and (9) are satisfied.

With regard to the polynomials of Eqs. (28–30), one can conclude that there may be more than 30 assembly configurations for a general member of this family of 3R1T parallel manipulators. However, the following section will show that these polynomials will be summarized to a minimal univariate polynomial of degree 12 for a special architecture of the manipulators of this class.

4. A SPECIAL ARCHITECTURE OF THE MANIPULATORS

Consider a case in which vectors $\overrightarrow{PB_1}$ and $\overrightarrow{PB_2}$ are collinear with vectors $\overrightarrow{PB_3}$ and $\overrightarrow{PB_4}$ respectively, so the moving platform becomes planar. Moreover, it is assumed that actuators are uniformly and symmetrically distributed around the passive leg and the first actuator is located on the xz-plane. Thus, in this case, we have

$$\mathbf{g} = -\mathbf{e} \tag{31a}$$

$$z_1 = z_3 = 0 \tag{31b}$$

$$a_{12} = a_{21} = a_{32} = a_{41} = 0 \tag{31c}$$

With the above conditions, the coefficients s_i $i \in \{2, 5, 10, 13, 14, 15, 16, 19, 20, 21\}$ become zero in Eqs. (10–13). Therefore, these equations will change to

$$s_1 e_1 + s_3 e_3 + s_4 = 0 \tag{32a}$$

$$s_6 f_2 + s_7 f_3 + s_8 = 0 \tag{32b}$$

$$s_9 e_1 + s_{11} e_3 - s_{12} = 0 \tag{32c}$$

$$s_{17}f_2+s_{18}f_3+s_{22}=0 \tag{32d}$$

Equations (32) constitute two systems of linear equations with respect to parameters (e_1, e_3) and (f_2, f_3) respectively which can be solved as

$$e_1=\frac{p_1}{p_3},\ e_3=\frac{p_2}{p_3}, \tag{33a}$$

$$f_2=\frac{p_4}{p_6}, f_3=\frac{p_5}{p_6}, \tag{33b}$$

where coefficients p_i (i=1,…, 6) are listed in appendix G. Introducing Eqs. (33) into Eqs. (8a), (8b) and (9a) results in

$$q_1e_2^2+q_2=0 \tag{34a}$$

$$q_3f_1^2+q_4=0 \tag{34b}$$

$$q_5e_2+q_6f_1+q_7=0 \tag{34c}$$

and coefficients q_i (i=1,…, 7) are listed in appendix H. Now, Eqs. (34) constitute a system of three nonlinear equations in three unknowns e_2, f_1 and h. Introducing Eqs. (34b) and (34c) into Eq. (34a) yields a linear polynomial in f_1 as follows

$$2q_1q_3q_6q_7f_1-q_1q_4q_6^2+q_1q_3q_7^2+q_2q_5^2q_3=0 \tag{35}$$

therefore

$$f_1=\frac{q_8}{q_9} \tag{36}$$

where

$$q_8=q_1q_4q_6^2-q_1q_3q_7^2-q_2q_5^2q_3$$

$$q_9=2q_1q_3q_6q_7$$

Now, the above value of f_1 is substituted into Eq. (34b) which gives

$$q_3q_8^2+q_4q_9^2=0 \tag{37}$$

Expanding Eq. (37) leads to a twelfth-degree polynomial in h as

$$\sum_{i=0}^{12} R_i h^i = 0 \tag{38}$$

The degree of above polynomial is much less then the degree of polynomial of Eq. (30), which is due to the special architecture of the manipulators. In the next example, it is shown that the above polynomial is minimal.

4.1. Numerical Example

In this example, the above presented method is applied to analyze the direct position kinematics of a 4RUS+1PS parallel manipulator meeting conditions (31). The schematic model of the manipulator is shown in Fig. 3.

For this manipulator, it is assumed that spherical joints of the active legs are located on a circle. Thus

$$b_1 = b_2 = b_3 = b \tag{39}$$

where b is the radius of the moving platform circle. Moreover, it is assumed that the revolute actuators are on a circle too. Therefore, vectors $\mathbf{a}_i$ can be computed from the following relations.

$$\begin{aligned} &\mathbf{a}_1 = [\, a + d_1 \cos\theta_1 \quad 0 \quad d_1 \sin\theta_1 \,]^T,\ \mathbf{a}_2 = [\, 0 \quad a + d_2 \cos\theta_2 \quad d_2 \sin\theta_2 \,]^T \\ &\mathbf{a}_3 = [\, -a + d_3 \cos\theta_3 \quad 0 \quad d_3 \sin\theta_3 \,]^T,\ \mathbf{a}_4 = [\, 0 \quad -a + d_4 \cos\theta_3 \quad d_4 \sin\theta_4 \,]^T \end{aligned} \tag{40}$$

where d_i is equal to $\overline{C_i A_i}$ and a is the radius of base circle. θ_i is the rotation angle of the i-th revolute actuator with respect to xy plane and is considered positive when counterclockwise. Now the direct position kinematics of the manipulator is solved with the followings values:

a=48 cm, b=40 cm, d_1=d_2=d_3=d_4=55 cm, l_1=90 cm, l_2=85 cm, l_3=105 cm, l_4=105 cm, ϕ_1=75°, θ_1=θ_2=65°, θ_3=θ_4=115°.

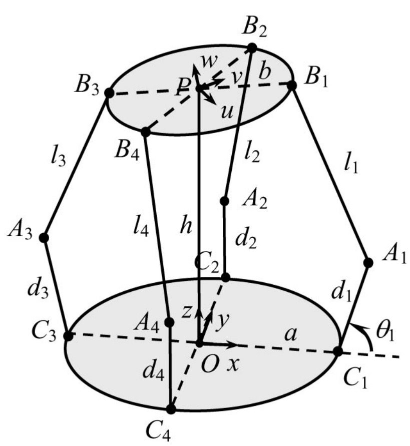

Fig. 3. The schematic model of a 4RUS+1PS parallel manipulator.

After calculating the parameters q_i and introducing them into Eq. (37), the coefficients R_i are calculated that are listed in Table 1. Now solving Eq. (38) yields twelve real solutions for h. For each value of h, the unique value of parameters e_i, f_i and consequently r_i (i=1, 2, 3) are obtained from Eqs. (33), (36), (34c) and (2). This leads to twelve real solutions for the direct position kinematics of the 4$\underline{\text{R}}$US+1PS manipulator that are presented in Table 2. Therefore, the univariate polynomial of Eq. (38) is minimal. The graphical representation of the solutions is also presented in Fig. 4. These solutions are correspondent to assembly modes of the manipulator.

In the next sections, direct velocity and direct kinematic singularities of the manipulators under study are analyzed. First, direct Jacobian matrix of the manipulators is obtained. However, for the sake of brevity, one of the manipulators of this family, the 4-RUS+1PS, is selected for this aim.

5. DIRECT JACOBIAN MATRIX

Differentiating both sides of Eq. (3) with respect to time yields

$$\underline{\mathbf{a}}_i + l_i\omega_i \times \mathbf{l}_i = \mathbf{v} + \omega \times \mathbf{b}_i \tag{41}$$

in which $\mathbf{v} = [\,v_x \quad v_y \quad v_z\,]^T$ and $\omega = [\,\omega_x \quad \omega_y \quad \omega_z\,]^T$ represent the three-dimensional linear and angular velocity of the moving platform, respectively. Moreover, $\boldsymbol{\omega}_i$ represents the three-dimensional angular velocity of link A_iB_i (Fig. 5). For the 4-RUS+1PS parallel manipulator, the vector $\underline{\mathbf{a}}_i$ can be written as

$$\underline{\mathbf{a}}_i = (\mathbf{n}_i \times \mathbf{d}_i)\dot{\theta}_i \tag{42}$$

where $\mathbf{n}_i$ represents a unit vector along the axis of i-th revolute actuator, $\dot{\theta}_i$ denotes the rate of this actuator and $\mathbf{d}_i$ is equal to vector $\overrightarrow{C_iA_i}$ (Fig. 5). Introducing Eq. (42) into Eq. (41) results in

$$(\mathbf{n}_i \times \mathbf{d}_i)\dot{\theta}_i + l_i\omega_i \times \mathbf{l}_i = \mathbf{v} + \omega \times \mathbf{b}_i \tag{43}$$

The passive variables $\boldsymbol{\omega}_i$ can be eliminated by dot multiplying both sides of Eq. (43) with $\mathbf{l}_i$, which gives

Table 1. Coefficients R_i (i=0,…, 12) obtained for the case study.

Coefficient	value	Coefficient	value
R_{12}	3.86878	R_5	$28555.4563 \times 10^{10}$
R_{11}	–2314.109686	R_4	$-42260.4576 \times 10^{11}$
R_{10}	564524.4070	R_3	$91230.8726 \times 10^{10}$
R_9	–70571452.93	R_2	$25120.6210 \times 10^{13}$
R_8	4476345931	R_1	$-42925.9208 \times 10^{13}$
R_7	$-91065.42992 \times 10^{6}$	R_0	$-47748.9411 \times 10^{13}$
R_6	$-47019.07760 \times 10^{8}$		

Table 2. Solutions for the direct position kinematics of the manipulator.

No.	h (cm)	(r_1, r_2, r_3)
1	80.2703	(–0.02781, –0.64099, 0.76704)
2	87.7983	(0.43036, –0.56197, 0.70638)
3	97.0392	(–0.50257, 0.26763, 0.82207)
4	100.4495	(0.44669, 0.28676, –0.84749)
5	106.5328	(–0.37533, –0.28944, –0.88054)
6	137.5899	(0.04949, 0.30219, 0.95196)
7	19.420	(0.73964, 0.26230, 0.61978)
8	2.660	(0.50257, –0.26762, 0.82207)
9	11.890	(–0.43036, 0.56197, 0.70638)
10	–0.770	(–0.44669, –0.28676, –0.84749)
11	–6.8399	(0.37533, 0.28944, –0.88054)
12	–37.8899	(–0.04949, –0.30219, 0.95196)

$$(\mathbf{n}_i \times \mathbf{d}_i)\cdot\mathbf{l}_i\dot{\theta}_i = \mathbf{v}\cdot\mathbf{l}_i + (\boldsymbol{\omega} \times \mathbf{b}_i)\cdot\mathbf{l}_i \tag{44}$$

where "·" denotes the dot product between vectors. Making use of formulae $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ and $\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}) = \mathbf{b}\cdot(\mathbf{c} \times \mathbf{a}) = \mathbf{c}\cdot(\mathbf{a} \times \mathbf{b})$, Eq. (44) can be rewritten as

$$(\mathbf{n}_i \times \mathbf{d}_i)\cdot\mathbf{l}_i\dot{\theta}_i = \mathbf{l}_i\cdot\mathbf{v} + (\mathbf{b}_i \times \mathbf{l}_i)\cdot\boldsymbol{\omega} \tag{45}$$

Let $\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{v}^T & \boldsymbol{\omega}^T \end{bmatrix}^T$ and $\dot{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 & \dot{\theta}_4 \end{bmatrix}^T$ be the vectors of the moving platform velocities and the actuated joint rates, respectively. Then, Eq. (45) can be written in the matrix form as

$$\mathbf{J}_q\dot{\boldsymbol{\theta}} = \mathbf{J}_x\dot{\mathbf{x}} \tag{46}$$

where

$$\mathbf{J}_q = \begin{bmatrix} (\mathbf{n}_1 \times \mathbf{d}_1)\cdot\mathbf{l}_1 & 0 & 0 & 0 \\ 0 & (\mathbf{n}_2 \times \mathbf{d}_2)\cdot\mathbf{l}_2 & 0 & 0 \\ 0 & 0 & (\mathbf{n}_3 \times \mathbf{d}_3)\cdot\mathbf{l}_3 & 0 \\ 0 & 0 & 0 & (\mathbf{n}_4 \times \mathbf{d}_4)\cdot\mathbf{l}_4 \end{bmatrix}_{4\times 4} \tag{47}$$

$$\mathbf{J}_x = \begin{bmatrix} \mathbf{l}_1^T & (\mathbf{b}_1 \times \mathbf{l}_1)^T \\ \mathbf{l}_2^T & (\mathbf{b}_2 \times \mathbf{l}_2)^T \\ \mathbf{l}_3^T & (\mathbf{b}_3 \times \mathbf{l}_3)^T \\ \mathbf{l}_4^T & (\mathbf{b}_4 \times \mathbf{l}_4)^T \end{bmatrix}_{4\times 6} \tag{48}$$

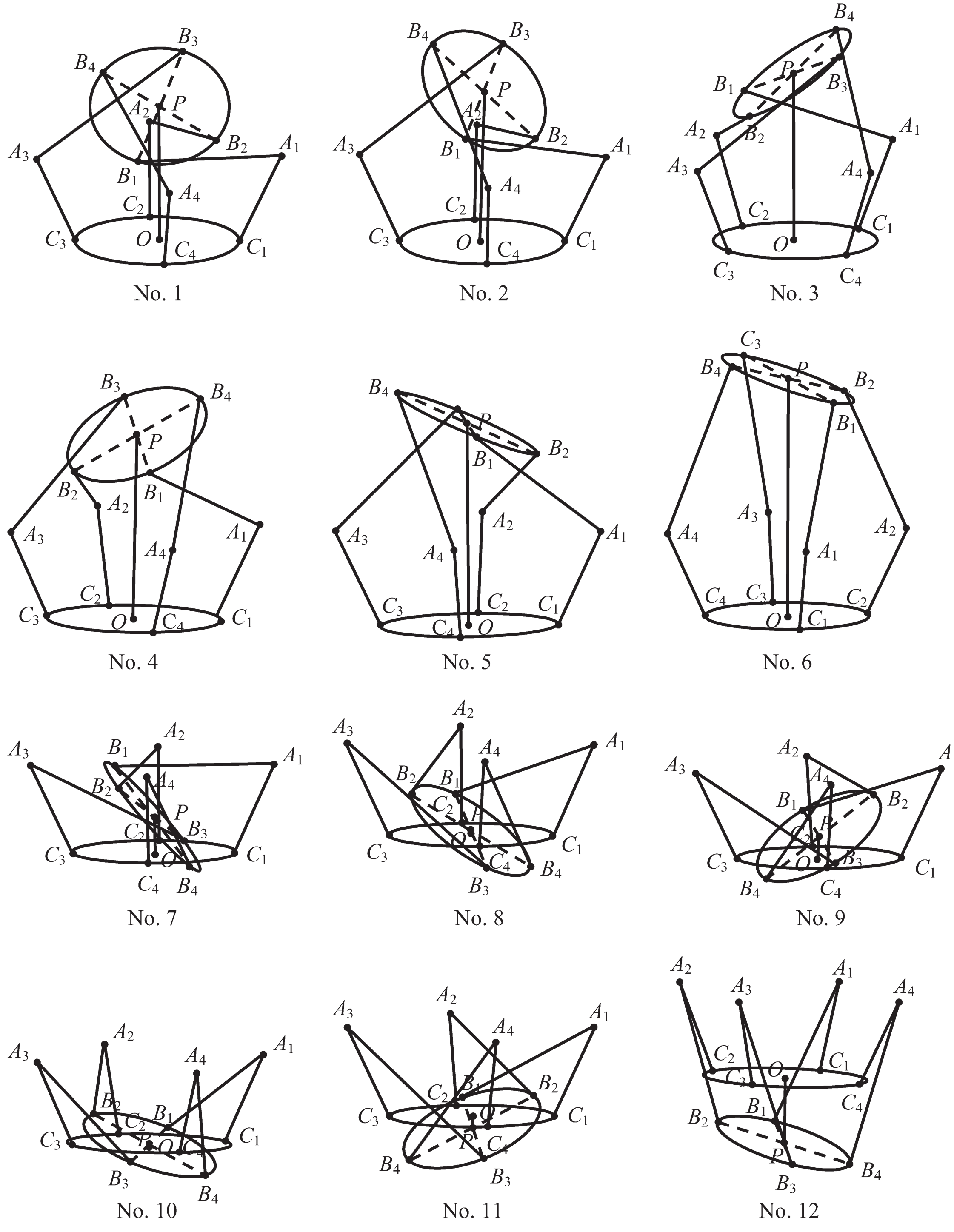

Fig. 4. Twelve solutions of the direct kinematics problem corresponding to assembly modes of the 4<u>R</u>US+1PS manipulator.

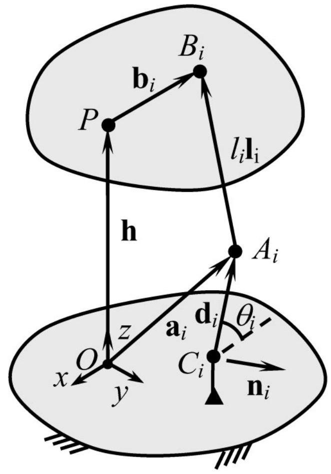

Fig. 5. Sketch of a typical leg of the 4-RUS+1PS parallel manipulator.

are the inverse and direct Jacobian matrices of the 4-RUS+1PS manipulator respectively. If we define $\mathbf{t}_i = \mathbf{b}_i \times \mathbf{l}_i$, $(i=1,\ldots,4)$ then matrix $\mathbf{J}_x$ can be written in compact form as follows

$$\mathbf{J}_x = \begin{bmatrix} \mathbf{l}_1^T & \mathbf{t}_1^T \\ \mathbf{l}_2^T & \mathbf{t}_2^T \\ \mathbf{l}_3^T & \mathbf{t}_3^T \\ \mathbf{l}_4^T & \mathbf{t}_4^T \end{bmatrix}_{4\times 6} \tag{49}$$

Applying the above procedure for other manipulators under study, one can conclude that Eq. (49) presents the direct Jacobian matrix of all of these manipulators. However, the inverse Jacobian matrices are different and depend on the types of the legs of the manipulators.

6. DIRECT VELOCITY ANALYSIS

The objective of direct velocity analysis is to determine the velocity of the moving platform from a given set of velocities of the actuators in a given pose.

It is obvious that, due to the constraining passive leg of the presented manipulators, we have $v_x = v_y = 0$; so the vector of moving platform velocity will be:

$$\dot{\mathbf{x}} = [0 \quad 0 \quad v_z \quad \omega_x \quad \omega_y \quad \omega_z]^T \tag{50}$$

which can be written in compact form as

$$\dot{\mathbf{x}}_c = [v_z \quad \omega_x \quad \omega_y \quad \omega_z]^T \tag{51}$$

Considering the above discussion, one can see that arrays of the first and second columns of $\mathbf{J}_x$ does not effect on the instantaneous kinematics of the manipulator, so these arrays can be eliminated and a new 4×4 direct Jacobian matrix is obtained as follows

$$\mathbf{J}_c = \begin{bmatrix} l_{1z} & \mathbf{t}_1^T \\ l_{2z} & \mathbf{t}_2^T \\ l_{3z} & \mathbf{t}_3^T \\ l_{4z} & \mathbf{t}_4^T \end{bmatrix}_{4\times 4} \tag{52}$$

where l_{iz} is the z component of vector $\mathbf{l}_i$. When $\mathbf{J}_c$ is invertible, Eq. (46) can be written as

$$\dot{\mathbf{x}}_c = \mathbf{J}_c^{-1}\mathbf{J}_q\dot{\boldsymbol{\theta}} \tag{53}$$

Equation (53) represents the direct velocity analysis of the parallel manipulators under study.

7. DIRECT KINEMATIC SINGULARITIES

A parallel manipulator gains extra degree(s) of freedom at direct kinematic singularities, even if all actuators are locked, and becomes difficult to control at such configurations. So these configurations should be found and avoided during the design and control stages of the manipulator. Numerous researchers have investigated singularity problems of parallel manipulators; see for instance [22–24]. In this section, direct kinematic singularities of the proposed 4-DOF parallel manipulators are identified based on rank deficiency of the direct Jacobian matrix $\mathbf{J}_c$ presented in Eq. (52). Since the matrix $\mathbf{J}_c$ is a square matrix, the direct kinematic singularities of the manipulators occur when det $(\mathbf{J}_c)=0$. Regarding Eq. (52), seven cases can be identified for the direct kinematic singularities.

Case (i) the first case in which direct kinematic singularities occurs is when all vectors $\mathbf{t}_i$ $(i=1,\ldots 4)$ are coplanar. Fig. 6a shows an example of this type of singularities in which the moving platform gains one rotational uncontrollable DOF(s) around the axis passing through point P and perpendicular to vectors $\mathbf{t}_i$.

Case (ii) the second case in which the direct kinematic singularities occurs is when the vectors $\mathbf{t}_i$ are parallel two by two. For example when

$$\mathbf{t}_1 || \mathbf{t}_3 \text{ and } \mathbf{t}_2 || \mathbf{t}_4 \tag{54}$$

These conditions are satisfied when the points (A_1, B_1, P, A_3, B_3) and (A_2, B_2, P, A_4, B_4) are located on two distinct planes, respectively. In this case, while all actuators are locked, the moving platform can rotate infinitesimally around the common line of the two planes, as shown in Fig. 6b. Therefore, manipulator gains one rotational uncontrollable DOF(s).

Case (iii) when the four vectors $\mathbf{t}_i$ $(i=1,\ldots, 4)$ are parallel with each other. This condition is satisfied when all vectors $\mathbf{b}_i$ and $\mathbf{l}_i$ $(i=1,\ldots, 4)$ are coplanar. An example of this type of singularities is depicted in Fig. 6c. In these singularities, the moving platform can rotate infinitesimally around any axis passing through point P and located on the plane of the points P, and B_i $(i=1,\ldots, 4)$. So, the manipulator gains two rotational uncontrollable DOFs around the u- and v- axes.

Case (iv) when arrays of the first column of matrix $\mathbf{J}_c$ are zero. This condition is satisfied when four vectors $\mathbf{l}_i$ $(i=1,\ldots,4)$ are parallel with the xy plane. In this case, the manipulator gains one translational uncontrollable DOF along the z-axis, Fig. 6d.

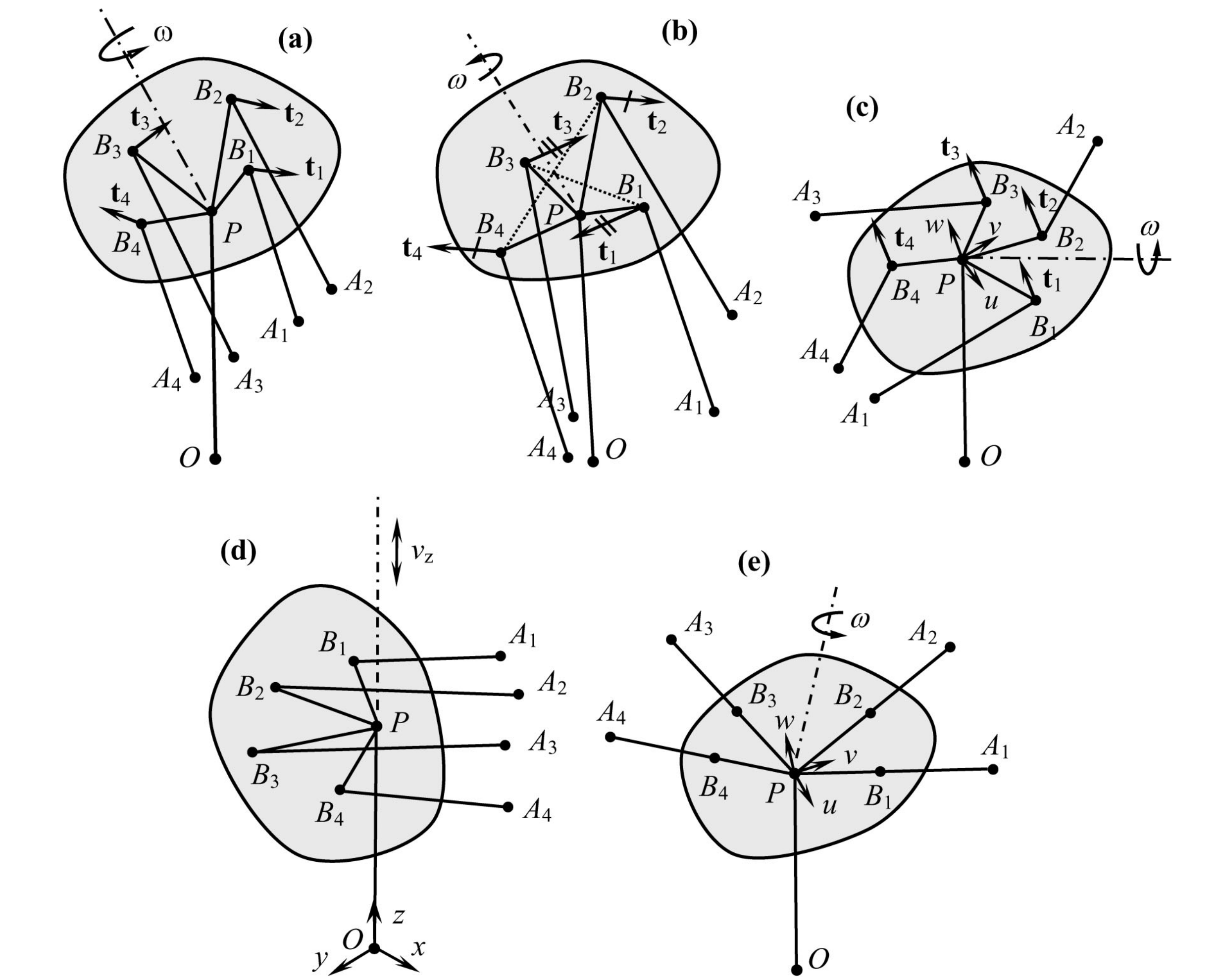

Fig. 6. Five cases of direct kinematic singularities of the manipulators: (a) when vectors $\mathbf{t}_i$ are coplanar, (b) when vectors $\mathbf{t}_i$ are parallel two by two, (c) when vectors $\mathbf{t}_i$ are parallel with each other, (d) when vectors $\mathbf{l}_i$ (i=1,…,4) are parallel with the xy plane and (e) when vectors $\mathbf{t}_i$ are zero.

Case (v) when the arrays of one of the other columns (rather than the first column) of matrix $\mathbf{J}_c$ are zero. This case is an special instance of case (i) in which vectors $\mathbf{t}_i$ (i=1,…,4) are coplanar and linearly dependent.

Case (vi) when the arrays of two columns (rather than the first column) of matrix $\mathbf{J}_c$ are zero. This case is an special instance of case (iii) in which vectors $\mathbf{t}_i$ (i=1,…,4) are parallel and linearly dependent.

Case (vii) when four vectors $\mathbf{t}_i$ (i=1,…, 4) are zero, i.e., when vectors $\mathbf{b}_i$ and $\mathbf{l}_i$ are collinear with each other for i=1,…, 4, as shown in Fig. 6e. In this case, the moving platform can rotate infinitesimally around any axis passing through point P, so the manipulator obtains three rotational uncontrollable DOFs around the u-, v- and w- axes.

8. CONCLUSION

A new family of 4-DOF 3R1T parallel manipulators with a passive constraining leg was presented which can be used for flight simulation. The paper describes topology generation process and shows four alternative structures depending on the forms of actuation. Using three unit vectors, the echelon form direct position analysis of the manipulators was performed. The analysis provides a univariate polynomial of degree 30 together with a set of two other univariate polynomials. A special case was also reported in which it is shown that the number and degrees of these polynomials seriously depend on architectures and they can reduce to a univariate polynomial of degree 12 for a special architecture of the manipulators. A numerical example was also included to show that the latter polynomial is minimal. Next, direct velocity of the manipulators was analyzed using Jacobian matrices. Finally, direct kinematic singularities were identified based on rank deficiently of the direct Jacobian matrix.

REFERENCES

1. Gough, V.E. and Whitehall, S.G., "Universal tyre test machine," *9th International Technical Congress*, FISITA, pp. 117–137, 1962.
2. Stewart, D.A., "Platform with six degrees of freedom," *Proceedings of the Institute of Mechanical Engineering*, Vol. 180, No. 5, pp. 371–386, 1965.
3. Clavel, R., "DELTA, A fast robot with parallel geometry," *Proceedings of 18th international symposium on industrial robots*, Lausanne, pp. 91–100, 1988.
4. Di Gregorio, R. and Parenti-Castelli, V., "A translational 3-DOF parallel manipulator," In: Lenarcic, J., Husty, M.L. editors. *Advances in robot kinematics:analysis and control, Dordrecht: Kluwer Academic Publishers*, pp. 49–58, 1998.
5. Chablat, D. and Wenger, P., "Architecture optimization of a 3-DOF translational parallel mechanism for machining applications, the Orthoglide," *IEEE Transactions on Robotics and Automation*, Vol. 19, No. 3, pp. 403–410, 2003.
6. Gallardo-Alvarado, J., Rico-Martínez, J.M., Alici, G.. "Kinematics and singularity analyses of a 4-dof parallel manipulator using screw theory," *Mechanism and Machine Theory*, Vol. 41, No. 9, pp. 1048–1061, 2006.
7. Bing, L. and Xiaojun, Y., Ying, H., "Kinematics analysis of a novel parallel platform with passive constraint chain," *International Journal of Design Engineering*, Vol. 1, No. 3, pp. 316–332, 2009.
8. Chen, W.J., "A novel 4-dof parallel manipulator and its kinematic modeling," *IEEE International Conference on Robotics and Automation*, Seoul, May, pp. 3350–3355, 2001.
9. Gosselin, C.M. and Angeles, J., "The optimum kinematic design of a spherical three-degree-of-freedom parallel manipulator," *ASME Journal of mechanisms, transmissions and automation in design*, Vol. 111, No. 2, pp. 202–207, 1989.
10. Huang, T., Gosselin, C.M., Whitehouse, D.J., Chetwynd, D.G., "Analytical approach for optimal design of a type of spherical parallel manipulator using dexterous performance indices," *Proceedings of the Institute of Mechanical Engineering, Part C: J. Mechanical Engineering Science*, Vol. 217, pp. 447–455, 2003.
11. Pouliot, N., Gosselin, C.M., Nahon, M., "Motion simulation capabilities of three-degree-of-freedom flight simulators," *AIAA Journal of Aircraft*, Vol. 35, No. 1, 9–17, 1998.
12. Koevermans, W.P. *et al.*, "Design and performance of the four d.o.f. motion system of the NLR research flight simulator," In *AGARD Conference Proceeding, No 198, Flight Simulation*, La Haye, pp. 17–1/17–11, 1975.
13. Zlatanov, D. and Gosselin, C.M., "A family of new parallel architectures with four degrees of freedom," *EJCK*, 1 (1), paper 6, 2002.

14. Zoppi, M., Zlatanov, D., Gosselin, C.M., "Analytical kinematics models and special geometries of a class of 4-DOF parallel mechanisms," *IEEE Transactions on Robotics*, Vol. 21, No. 6, pp. 1046–1055, 2005.
15. Kong, X. and Gosselin, C. M., "Type synthesis of 4-DOF SP-equivalent parallel manipulators: A virtual chain approach," *Mechanism and Machine Theory*, Vol. 41, No. 11, pp. 1306–1319, 2006.
16. Hervé, J. M., "Analyse structurelle des mécanismes par groupe des déplacements," *Mechanism and Machine Theory*, Vol. 13, pp. 437–450, 1978.
17. Fang, Y. and Tsai, L.-W., "Structure synthesis of a class of 4-DoF and 5-DoF parallel manipulators with identical limb structures," *The International Journal of Robotics Research,* Vol. 21, No. 9, pp. 799–810, 2002.
18. Kong, X. and Gosselin, C.M., "Type synthesis of Three-degree-of-freedom spherical parallel manipulators," *The International Journal of Robotics Research*, Vol. 23, No. 3, pp. 237–245, 2004.
19. Carricato, M. and Parenti-Castelli, V., "A family of 3-DOF translational parallel Manipulators," *ASME Journal of Mechanical Design*, Vol. 125, No. 2, pp. 302–307, 2003.
20. Brogardh, T., "PKM Research - Important Issues, as seen from a Product Development Perspective at ABB Robotics," In Proceedings of the WORKSHOP on *"Fundamental Issues and Future Research Directions for Parallel Mechanisms and Manipulators*" October 3–4, Quebec, Canada, pp. 68–82, 2002.
21. Salmon, D. D., G., *Lessons introductory to the modern higher algebra,* third edition, Adamant Media Corporation, 2002.
22. Gosselin, C. M. and Angeles, J., "Singularity analysis of closed-loop kinematic chains," *IEEE Transactions on Robotics and Automation*, Vol. 6, No. 3, pp. 281–90, 1990.
23. Zlatanov, D., Fenton, R.G., Benhabib, B., "Singularity Analysis of Mechanisms and Robots Via a Velocity-Equation Model of The Instantaneous Kinematics," *Proceedings of IEEE international conference on robotics and automation*, San Diego, USA,Vol. 2, pp. 986 –991, 1994.
24. Zhao, J-S, Feng, Z-J, Zhou, K., Dong, J-X, "Analysis of the singularity of spatial parallel manipulator with terminal constraints," *Mechanism and Machine Theory*, Vol. 40, No. 3, pp. 275 –284, 2005.

APPENDIX A

Coefficients s_i are defined as

$$s_1 = -2a_{11}b_1, s_2 = -2a_{12}b_1, s_3 = (2h - 2a_{13})b_1, s_4 = b_1^2 + a_{12}^2 + a_{11}^2 + a_{13}^2 - 2ha_{13} - l_1^2 + h^2$$

$$s_5 = -2a_{21}b_2, s_6 = -2a_{22}b_2, s_7 = (2h - 2a_{23})b_2, s_8 = b_2^2 + a_{22}^2 + a_{21}^2 + a_{23}^2 - 2ha_{23} - l_2^2 + h^2,$$

$$s_9 = -2a_{31}b_3, s_{10} = -2a_{32}b_3, s_{11} = (2h - 2a_{33})b_3, s_{12} = b_3^2 - 2ha_{33} + a_{32}^2 + a_{31}^2 + a_{33}^2 - l_3^2 + h^2,$$

$$s_{13} = -2z_1a_{41}b_1$$

$$s_{15} = 2z_1b_1h - 2z_1b_1a_{43}, s_{16} = -2z_2a_{41}b_2, s_{17} = -2z_2a_{42}b_2$$

$$s_{18} = 2z_2b_2h - 2z_2a_{43}b_2, s_{19} = -2z_3a_{41}b_3, s_{20} = -2z_3a_{42}b_3, s_{21} = 2z_3b_3h - 2z_3a_{43}b_3$$

$$s_{22} = h^2 - l_4^2 + a_{41}^2 + a_{42}^2 - 2ha_{43} + a_{43}^2 + z_1^2b_1^2 + z_2^2b_2^2 + z_3^2b_3^2 +$$

$$2b_1b_3z_1z_3\delta_2 + 2b_2b_3z_2z_3\delta_3 + 2b_1b_2z_1z_2\delta_1$$

APPENDIX B

Coefficients t_i are

$$t_1 = -s_2 s_{15} s_5 s_9 + s_3 s_5 s_9 s_{14},\ t_2 = s_2 s_{16} s_9 s_6 - s_2 s_{17} s_5 s_9,\ t_3 = s_2 s_{16} s_9 s_7 - s_2 s_{18} s_5 s_9$$

$$t_4 = (-s_2 s_{20} s_5 s_9 + s_2 s_{19} s_5 s_{10}) g_2 + (s_2 s_{19} s_5 s_{11} - s_2 s_{21} s_5 s_9) g_3 +$$

$$s_4 s_5 s_9 s_{14} + s_2 s_{16} s_9 s_8 - s_2 s_{22} s_5 s_9 + s_2 s_{19} s_5 s_{12}$$

$$t_5 = s_5 s_9 (s_{13} s_2 - s_{14} s_1),\ t_6 = -s_{13} s_5 s_9 s_3 + s_{15} s_1 s_5 s_9,\ t_7 = s_{17} s_1 s_5 s_9 - s_{16} s_1 s_9 s_6$$

$$t_8 = -s_{16} s_1 s_9 s_7 + s_{18} s_1 s_5 s_9$$

$$t_9 = (-s_{19} s_1 s_5 s_{10} + s_{20} s_1 s_5 s_9) g_2 + (s_{21} s_1 s_5 s_9 - s_{19} s_1 s_5 s_{11}) g_3 +$$

$$s_{22} s_1 s_5 s_9 - s_{13} s_5 s_9 s_4 - s_{16} s_1 s_9 s_8 - s_{19} s_1 s_5 s_{12}$$

APPENDIX C

Coefficients u_i and G_i are as follows

$$u_1 = t_6^2 + t_5^2 + t_1^2,\ u_2 = t_7^2 + t_2^2,\ u_3 = t_3^2 + t_8^2,\ u_4 = 2t_1 t_2 + 2t_6 t_7,\ u_5 = 2t_6 t_8 + 2t_1 t_3$$

$$u_6 = 2t_7 t_8 + 2t_2 t_3,\ u_7 = 2t_6 t_9 + 2t_1 t_4,\ u_8 = 2t_7 t_9 + 2t_2 t_4,\ u_9 = 2t_8 t_9 + 2t_3 t_4,$$

$$u_{10} = -t_5^2 + t_9^2 + t_4^2,\ u_{11} = s_5^2 + s_6^2,\ u_{12} = s_7^2 + s_5^2, u_{13} = 2s_6 s_7,\ u_{14} = 2s_6 s_8$$

$$u_{15} = 2s_7 s_8,\ u_{16} = -s_5^2 + s_8^2,\ u_{17} = -t_2 s_6 + s_5 t_7,\ u_{18} = -t_3 s_7,\ u_{19} = s_5 t_6 - t_1 s_6$$

$$u_{20} = -t_1 s_7 + t_5 s_5,\ u_{21} = s_5 t_8 - t_2 s_7 - t_3 s_6,\ u_{22} = -t_1 s_8$$

$$u_{23} = -t_4 s_6 - t_2 s_8 + s_5 t_9,\ u_{24} = -t_3 s_8 - t_4 s_7,\ u_{25} = -t_4 s_8 - \delta_1 t_5 s_5$$

$$u_{26} = (-t_1 s_{10} + s_9 t_6) g_2 + (t_5 s_9 - t_1 s_{11}) g_3 - t_1 s_{12},\ u_{27} = (-t_2 s_{10} + s_9 t_7) g_2 - t_2 s_{11} g_3 - t_2 s_{12}$$

$$u_{28}=(s_9t_8-t_3s_{10})g_2-t_3s_{11}g_3-t_3s_{12},\ u_{29}=(-t_4s_{10}+s_9t_9)g_2-t_4s_{11}g_3-\delta_2t_5s_9-t_4s_{12}$$

$$u_{30}=(s_6s_{10}+s_9s_5)g_2+s_6s_{11}g_3+s_6s_{12},\ u_{31}=s_7s_{10}g_2+(s_7s_{11}+s_9s_5)g_3+s_7s_{12}$$

$$u_{32}=s_8s_{10}g_2+s_8s_{11}g_3-\delta_3s_9s_5+s_8s_{12}$$

$$G_1=s_9^2+s_{10}^2,\ G_2=2(s_{10}s_{12}+s_{10}s_{11}g_3),\ G_3=(s_{11}^2+s_9^2)g_3^2+2s_{11}s_{12}g_3-s_9^2+s_{12}^2$$

APPENDIX D

Coefficients v_i, w_i and m_i are as follows

$$v_1=-u_{26}u_{31},\ v_2=u_{28}u_{32}-u_{29}u_{31},\ v_3=-u_{30}u_{28}+u_{31}u_{27},\ v_4=u_{30}u_{26},\ v_5=u_{30}u_{29}-u_{32}u_{27}$$

$$w_1=u_5v_3v_4+u_1v_3^2+u_3v_4^2+u_2v_1^2+u_6v_1v_4+u_4v_3v_1$$

$$w_2=u_6v_2v_4+2u_3v_4v_5+2u_2v_1v_2+u_4v_3v_2+u_5v_3v_5+u_6v_1v_5+u_9v_3v_4+u_7v_3^2+u_8v_3v_1$$

$$w_3=u_8v_3v_2+u_6v_2v_5+u_3v_5^2+u_2v_2^2+u_{10}v_3^2+u_9v_3v_5$$

$$w_4=u_{11}v_1^2+u_{12}v_4^2+u_{13}v_1v_4$$

$$w_5=2u_{12}v_4v_5+u_{14}v_3v_1+u_{13}v_1v_5+u_{13}v_2v_4+2u_{11}v_1v_2+u_{15}v_3v_4$$

$$w_6=u_{15}v_3v_5+u_{16}v_3^2+u_{11}v_2^2+u_{14}v_3v_2+u_{12}v_5^2+u_{13}v_2v_5$$

$$w_7=u_{17}v_1^2+u_{18}v_4^2+u_{19}v_3v_1+u_{20}v_3v_4+u_{21}v_1v_4$$

$$w_8=2u_{18}v_4v_5+2u_{17}v_1v_2+u_{19}v_3v_2+u_{21}v_2v_4+u_{20}v_3v_5+u_{21}v_1v_5+u_{24}v_3v_4+u_{22}v_3^2+u_{23}v_3v_1$$

$$w_9 = u_{21} v_2 v_5 + u_{18} v_5^2 + u_{17} v_2^2 + u_{23} v_3 v_2 + u_{25} v_3^2 + u_{24} v_3 v_5$$

$$m_1 = w_2 w_4 - w_1 w_5, m_2 = w_3 w_4 - w_1 w_6, m_3 = w_2 w_7 - w_1 w_8, m_4 = w_3 w_7 - w_1 w_9$$

APPENDIX E

The term g_2^4 can be eliminated from Eqs. (17) and (26) through multiplying them by $H_1 g_2^2$ and G_1 respectively. Subtraction of the obtained expressions results in

$$(G_1 H_2 - G_2 H_1) g_2^3 + (G_1 H_3 - G_3 H_1) g_2^2 + G_1 H_4 g_2 + G_1 H_5 = 0 \tag{E1}$$

The procedure is repeated by multiplying Eqs. (17) and (26) in $(H_1 g_2 + H_2) g_2^2$ and $G_1 g_2 + G_2$ respectively. Subtraction of the obtained equations yields

$$(G_1 H_3 - G_3 H_1) g_2^3 + (G_1 H_4 + G_2 H_3 - G_3 H_2) g_2^2 + (G_1 H_5 + G_2 H_4) g_2 + G_2 H_5 = 0 \tag{E2}$$

In addition, multiplying Eq. (17) by g_2 results in

$$G_1 g_2^3 + G_2 g_2^2 + G_3 g_2 = 0 \tag{E3}$$

Eqs. (E1) −(E3) and (17) can be written in a matrix form as

$$\mathbf{K}_1 \begin{bmatrix} g_2^3 \\ g_2^2 \\ g \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{E4}$$

where

$$\mathbf{K}_1 = \begin{bmatrix} G_1 H_2 - G_2 H_1 & G_1 H_3 - G_3 H_1 & G_1 H_4 & G_1 H_5 \\ G_1 H_3 - G_3 H_1 & G_1 H_4 + G_2 H_3 - G_3 H_2 & G_1 H_5 + G_2 H_4 & G_2 H_5 \\ G_1 & G_2 & G_3 & 0 \\ 0 & G_1 & G_2 & G_3 \end{bmatrix}$$

Eq. (E4) is valid if and only if $\det(\mathbf{K}_1) = 0$. Equating determinant of $\mathbf{K}_1$ to zero leads to a polynomial of degree 16 as Eq. (28).

APPENDIX F

Taking $(17) \times H_6 - (27) \times G_1$ and $(17) \times H_8 - (27) \times G_3$ respectively and rewriting the resultant equations in matrix form, we have

$$\mathbf{K}_2 \begin{bmatrix} g_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{F1}$$

where

$$\mathbf{K}_2 = \begin{bmatrix} G_2H_6 - G_1H_7 & G_3H_6 - G_1H_8 \\ G_1H_8 - G_3H_6 & G_2H_8 - G_3H_7 \end{bmatrix}$$

Again, Eq. (F1) is valid if and only if $\det(\mathbf{K}_2)=0$. Thus

$$(G_2H_6 - G_1H_7)(G_2H_8 - G_3H_7) + (G_3H_6 - G_1H_8)^2 = 0 \tag{F2}$$

Now, taking into account the values of parameters H_i and G_i, Eq. (F2) can be written as Eq. (29).

APPENDIX G

Coefficients p_i are as follows

$$p_1 = s_3s_{12} + s_4s_{11},\ p_2 = -s_4s_9 - s_1s_{12},\ p_3 = s_3s_9 - s_1s_{11},$$

$$p_4 = s_7s_{22} - s_8s_{18},\ p_5 = s_8s_{17} - s_6s_{22},\ p_6 = s_6s_{18} - s_7s_{17}$$

APPENDIX H

Coefficients q_i are

$$q_1 = p_3^2, q_2 = p_1^2 + p_2^2 - p_3^2, q_3 = p_6^2, q_4 = p_4^2 + p_5^2 - p_6^2, q_5 = p_3p_4, q_6 = p_1p_6, q_7 = p_2p_5 - p_3^3p_6$$