DESIGN AND ANALYSIS OF CAM MECHANISM WITH TRANSLATING FOLLOWER HAVING DOUBLE ROLLERS

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ABSTRACT

The pressure angle is one of the primary considerations in designing a cam mechanism since an inappropriate angle may cause excessive sliding loads on the follower. This paper presents a simple yet straightforward method for the design and analysis of a cam mechanism with a translating follower having double rollers. In the proposed approach, conjugate surface theory is employed to derive a kinematic model of the cam mechanism. Analytical expressions for the pressure angle and principal curvatures of the cam profile are then derived. The validity of the analytical expressions is confirmed by machining a designed cam using a 3-axis CNC machine tool.

Keywords: homogenous coordinate transformation; double roller; pressure angle.

CONCEPTION ET ANALYSE D'UN MÉCANISME À CAME AVEC SUIVEURS TRANSLATIONELS À DOULE ROULEAUX

RÉSUMÉ

L'angle de pression est une des premières considérations dans la conception d'un mécanisme à came, étant donné qu'un angle inapproprié peut causer une charge glissante excessive sur le suiveur. Cet article présente une méthode simple et directe pour la conception et l'analyse d'un mécanisme à came avec suiveur translationel à double rouleaux. Dans l'approche proposée, la théorie de la surface conjuguée est employée pour donner un modèle cinématique du mécanisme à came. Les expressions analytiques pour l'angle de pression et les courbes principales du profil de la came sont dérivées. La validité des expressions analytiques est confirmée par la fabrication d'une came à l'aide d'une machine-outil CNC.

Mots-clés : transformation homogène coordonnée; double rouleaux; angle de pression.

NOMENCLATURE

$(xyz)_0$	Configuration frame $(xyz)_0$ built in cam
${}^0\mathbf{S}$	Cam surface
${}^r\mathbf{S}$	Roller surface
${}^0\mathbf{n}$	Unit outward normal of cam surface
ρ	Roller radius
R_0	Pitch circle radius of cam
h	Life height of follower
e	Offset amount of roller
ψ	Pressure angle
θ_1	Rotation angle of cam
s_2	Displacement of follower

1. INTRODUCTION

Cam elements have many advantageous features for mechanical transmission systems, including simplicity, reliability, repeatability and a low running noise. However, the pressure angle must be carefully designed since an overly-large angle may result in an excessive sliding load on the follower [1]. In practice, the pressure angle can be reduced by increasing the offset amount of the follower. However, if the offset is specified in such a way as to reduce the pressure angle during the rise stage of the follower, the pressure angle during the fall stage inevitably increases, and vice versa [2]. Hence, a careful design of the offset amount is essential in achieving the optimal compromise. In most practical cam systems, the follower is fitted with a spring which presses it firmly against the cam surface as it rotates. However, Wunderlich [3] proposed a single-disk cam mechanism in which vibration under high-speed conditions was minimized by replacing the spring with an oscillating double roller follower. Duca et al. [4] performed an analytical investigation to establish the design parameters which minimized the pressure angle for the cam mechanism proposed in [3]. Wu et al. [5] presented a disc cam mechanism with a novel translating follower incorporating two double rollers mounted symmetrically on opposite sides of the cam. In developing the proposed mechanism, the instantaneous center of velocity principle was applied to design the cam profile in such a way as to minimize the pressure angle during both the rise stage and the fall stage of the follower motion. Hsieh [6] used a homogeneous coordinate transformation method to derive an analytical expression for the curved slots of a Geneva wheel cam mechanism. A method was then proposed for deriving the corresponding NC machining data.

However, the published literature lacks a systematic investigation into the design and machining of a disc cam mechanism with a translating double roller follower. Accordingly, this study presents a simple yet robust methodology for the design, analysis and machining of such mechanisms. The proposed methodology comprises four steps, namely (1) establishing a kinematic model of the cam mechanism, (2) designing the cam profile, (3) analyzing the pressure angle and principal curvatures of the cam, and (4) generating the NC data required to machine the designed cam. The validity of the proposed methodology is demonstrated by machining a designed cam on a 3-axis CNC machine.

2. SURFACE GEOMETRY

Figure 1 illustrates the cam mechanism considered in the present study. As shown, the translating follower incorporates two identical rollers, denoted by the suffixes $r1$ and $r2$, respectively. The rollers are mounted symmetrically on opposite sides of the follower centerline and have the same offset. As shown in the figure, the cam rotates in the clockwise direction about its center point, 0_0. The profile of the cam comprises four segments, each designed to produce a specific follower motion, namely (1) follower rise (segment p_1p_2),

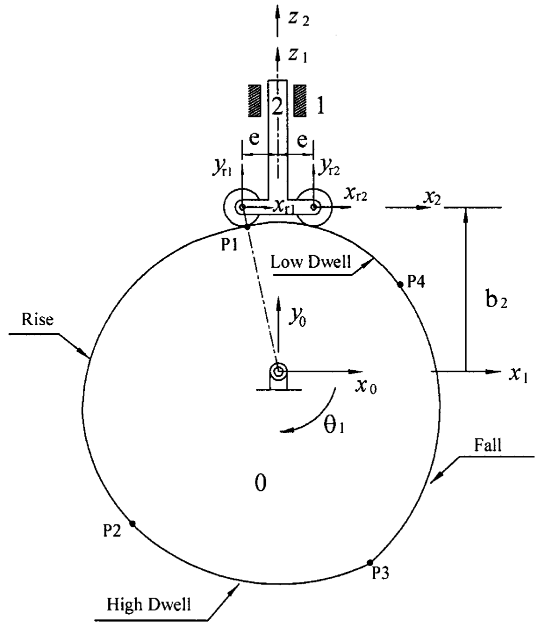

Fig. 1. Cam mechanism with translating follower having double rollers.

Table 1. Kinematic parameters of cam mechanism with translating follower having double rollers.

	b_i	θ_i	a_i	α_i
Link 1	0	θ_1	0	$-90°$
Link 2	b_2	0	0	0

(2) follower dwell (segment p_2p_3), (3) follower fall (segment p_3p_4), and (4) follower dwell (segment p_4p_0). Figure 2 shows a typical double-dwell cam profile. As the cam rotates, the follower rises to a total lift height h over an angle θ_r, remains in a high dwell position for the next θ_H, falls through a distance h over θ_f, and remains in a low dwell position over θ_L. As shown, at least one of the rollers remains in contact with the cam surface at all times as the cam rotates.

To determine the cam profile using conjugate theory, it is first necessary to label the coordinate frames, starting with the cam (marked as “0” in Fig. 1) and ending with the translating follower (marked as “2” in Fig. 1). Once a frame $(xyz)_i (i = 0,1,2)$ has been assigned to every link according to the Denavit–Hartenberg (D–H) notation [7], the kinematic parameters can be tabulated, as shown in Table 1. Note that parameter b_2 is defined as $b_2 = b_{20} + s_2(\theta_1)$, where b_{20} is the shortest distance between the cam center and the center of the follower rollers as the cam rotates and s_2 is the translating follower displacement.

The configuration of frame $(xyz)_2$ with respect to frame $(xyz)_0$ is given by

$$ {}^0\mathbf{A}_2 = \prod_{i=1}^{2} {}^{i-1}\mathbf{A}_i = \begin{bmatrix} C\theta_1 & 0 & -S\theta_1 & -b_2 S\theta_1 \\ S\theta_1 & 0 & C\theta_1 & b_2 C\theta_1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{1} $$

where C and S denote COSINE and SINE, respectively.

To analyze the roller contact with the cam during cam rotation, it is first necessary to establish the frames of the two rollers, namely $(xyz)_{r1}$ and $(xyz)_{r2}$, respectively. The configuration of frame $(xyz)_{r1}$ with respect to frame $(xyz)_2$ is given by the matrix

$$
{}^2\mathbf{A}_{r1} = \text{Trans}(-e,0,0)\text{Rot}(x_2,90) = \begin{bmatrix} 1 & 0 & 0 & -e \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2}
$$

where e is the offset amount of two rollers.

Similarly, the configuration of frame $(xyz)_{r2}$ with respect to frame $(xyz)_2$ is given by

$$
{}^2\mathbf{A}_{r2} = \text{Trans}(e,0,0)\text{Rot}(x_2,90) = \begin{bmatrix} 1 & 0 & 0 & e \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3}
$$

The two rollers are identical. Thus, the surface equation, ${}^rS$, and unit outward normal, ${}^rn$, are also the same for both rollers, and can be expressed with respect to frame $(xyz)_r$ as follows:

$$
{}^r\mathbf{S} = [\rho C\theta \quad \rho S\theta \quad u \quad 1]^T \quad (-\lambda/2 \le u \le \lambda/2, 0 \le \theta \le 2\pi), \tag{4}
$$

$$
{}^r\mathbf{n} = \frac{\partial^r\mathbf{S}}{\partial u} \times \frac{\partial^r\mathbf{S}}{\partial \theta} \bigg/ \left| \frac{\partial^r\mathbf{S}}{\partial u} \times \frac{\partial^r\mathbf{S}}{\partial \theta} \right| = [C\theta \quad S\theta \quad 0 \quad 0]^T, \tag{5}
$$

where θ is the polar angle, u is the height parameter of the cylindrical roller, and ρ is the radius of the roller.

In accordance with the principles of homogeneous coordinate transformation, the cam surface equation, ${}^0\mathbf{S}$, is related to rollers $r1$ and $r2$ as follows:

$$
{}^0\mathbf{S} = {}^0\mathbf{A}_2\, {}^2\mathbf{A}_{r1}\, {}^{r1}\mathbf{S} = \begin{bmatrix} \rho C(\theta+\theta_1) - eC\theta_1 - b_2 S\theta_1 \\ \rho S(\theta+\theta_1) - eS\theta_1 + b_2 C\theta_1 \\ u \\ 1 \end{bmatrix}, \tag{6}
$$

$$
{}^0\mathbf{S} = {}^0\mathbf{A}_2\, {}^2\mathbf{A}_{r1}\, {}^{r1}\mathbf{S} = \begin{bmatrix} \rho C(\theta+\theta_1) + eC\theta_1 - b_2 S\theta_1 \\ \rho S(\theta+\theta_1) + eS\theta_1 + b_2 C\theta_1 \\ u \\ 1 \end{bmatrix}. \tag{7}
$$

In addition, the normal vector, ${}^0\mathbf{n}$, of the cam can be expressed as

$$
{}^o\mathbf{n} = {}^0\mathbf{A}_r\, {}^r\mathbf{n} = [C(\theta+\theta_1)\ S(\theta+\theta_1) \quad 0 \quad 0]^T. \tag{8}
$$

Once the input-output relation of the cam mechanism has been defined, the conjugate points and cam profiles can be determined via the formula

$$
{}^0\mathbf{n}^T \cdot \frac{d^0\mathbf{S}}{dt} = 0, \tag{9}
$$

where ${}^0\mathbf{n}$ and ${}^0\mathbf{S}$ are the unit outward normal and surface equation of the cam profile with respect to frame $(xyz)_0$, respectively. The conjugate point (denoted as $\bar{\theta}$) is expressed as

$$
\bar{\theta} = -\tan^{-1}\left(\frac{\bar{F}}{\bar{E}}\right), \tag{10}
$$

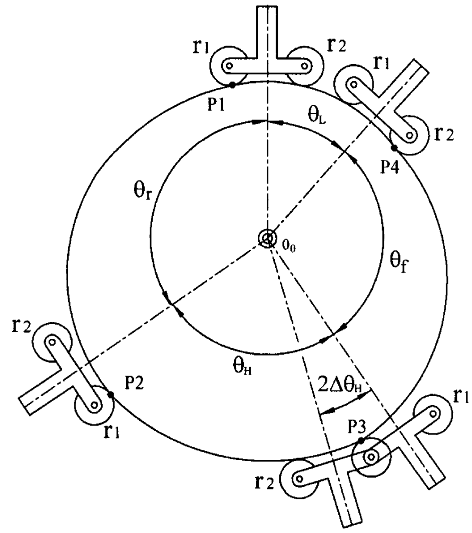

Fig. 2. Cam profile corresponding to roller contact position.

where $\bar{E}$ is defined as $\bar{E} = b_2$, and $\bar{F}$ is defined as $\bar{F} = e + db_2/d\theta_1$ for roller $r1$ and $\bar{F} = -e + db_2/d\theta_1$ for roller $r2$.

Consequently, the profiles of the cam corresponding to rollers $r1$ and $r2$ can be expressed respectively as

$$ {}^0\mathbf{S} = {}^0\mathbf{A}_{r1}\ {}^{r1}\mathbf{S} = \begin{bmatrix} \rho C(\bar{\theta}+\theta_1) - eC\theta_1 - b_2 S\theta_1 \\ \rho S(\bar{\theta}+\theta_1) - eS\theta_1 + b_2 C\theta_1 \\ u \\ 1 \end{bmatrix}, \tag{11} $$

$$ {}^0\mathbf{S} = {}^0\mathbf{A}_{r1}\ {}^{r1}\mathbf{S} = \begin{bmatrix} \rho C(\bar{\theta}+\theta_1) + C\theta_1 - b_2 S\theta_1 \\ \rho S(\bar{\theta}+\theta_1) + eS\theta_1 + b_2 C\theta_1 \\ u \\ 1 \end{bmatrix}. \tag{12} $$

As shown in Fig. 2, the cam profile is determined by roller $r1$ (i.e., Eq. (11)) over the interval $0 \leq \theta_1 \leq (\theta_r + \theta_H - 2\Delta\theta_H)$ and by roller $r2$ (i.e., Eq. (12)) over the interval $(\theta_r + \theta_H) \leq \theta_1 \leq 360°$.

As discussed above, the low and high dwell motions of the follower correspond to two particular segments (circular arcs) of the cam profile. For the low dwell interval, the circular cam segment has a radius of R_0, i.e., the base circle radius of the cam. By contrast, for the high dwell interval, the radius R of the circular arc is given by

$$ R = \sqrt{(b_{20}+h)^2 + e^2} - \rho. \tag{13} $$

In the high dwell interval, both of the rollers contact the cam profile. Thus, half the subtending angle of the roller span, $\Delta\theta_H$ can be expressed as

$$ \Delta\theta_H = \sin^{-1}\left(\frac{e}{R+\rho}\right). \tag{14} $$

The same condition also applies in the low dwell interval. As a result, half the subtending angle of the roller span, $\Delta\theta_L$, can be expressed as

$$ \Delta\theta_L = \sin^{-1}\left(\frac{e}{R_0+\rho}\right). \tag{15} $$

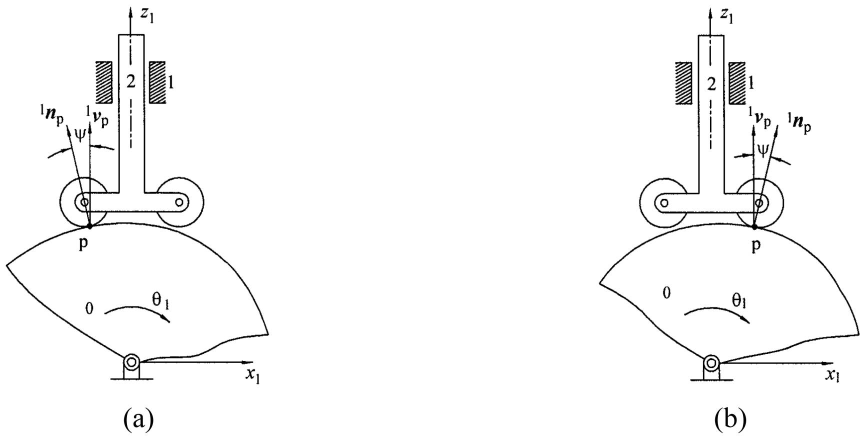

Fig. 3. Pressure angle of cam mechanism during: (a) rise of follower and (b) fall of follower.

It is noted that to satisfy the conditions given in Eqs. (14) and (15), the two circular arcs corresponding to the high and low dwell intervals, respectively, must be sufficiently large to accommodate the roller span.

3. PRESSURE ANGLE

The pressure angle, ψ, provides an important measure of the effectiveness with which a force is transmitted between a cam and a follower. Specifically, a pressure angle of zero degrees maximizes the transmitted force, while a pressure angle of 90 degrees results in a jamming of the follower in its guides. As shown in Fig. 3, the pressure angle is defined as the angle between the direction of motion (i.e., velocity) of the follower and the direction of the force transmission axis [8]. The pressure angle at the contact point p between the cam and the roller is given as

$$\tan\psi = \frac{||{}^1\mathbf{n}_p \times {}^1\mathbf{v}_p||}{|{}^1\mathbf{n}_p \cdot {}^1\mathbf{v}_p|}, \tag{16}$$

where the unit normal vector of the contact point, ${}^1\mathbf{n}_p$, and the direction of velocity of the contact point, ${}^1\mathbf{v}_p$, are both defined with respect to the fixed frame $(xyz)_1$.

The unit normal vector of the contact point on the roller-follower surface with respect to frame $(xyz)_1$ is given by

$${}^1\mathbf{n}_p = {}^1\mathbf{A}_2\, {}^2\mathbf{A}_r\, {}^r\mathbf{n}_p = [C\bar{\theta} \quad 0 \quad S\bar{\theta} \quad 0]^T. \tag{17}$$

Meanwhile, the location of the contact point, $\mathbf{p}$, on the roller-follower surface with respect to frame $(xyz)_1$ is given by

$${}^1\mathbf{S}_p = {}^1\mathbf{A}_2\, {}^2\mathbf{A}_r\, {}^r\mathbf{S}_p = [\rho C\bar{\theta} - e \quad -u \quad \rho S\bar{\theta} + b_2 \quad 1]^T. \tag{18}$$

Since the roller-follower translates along the z_1 axis, the direction of velocity of the contact point on the roller-follower surface with respect to frame $(xyz)_1$ can be expressed as

$${}^1\mathbf{v}_p = \frac{d\,{}^1\mathbf{S}}{dt} = \begin{bmatrix} 0 & 0 & \dfrac{db_2}{dt} & 0 \end{bmatrix}^T. \tag{19}$$

Using the inner product and cross product, the pressure angle, ψ, can be derived as

$$\tan\psi = |\cot\bar{\theta}|. \tag{20}$$

In determining the pressure angle, ψ, as the cam rotates, it is necessary to consider the contact status between the two rollers and the cam profile. More specifically:

1. For $0 \leq \theta_1 \leq \theta_r + 2\Delta\theta_H$, only roller $r1$ contacts the cam profile, and the pressure angle can be derived from the conjugate angle $\bar{\theta}$ using Eq. (20).

2. For $\theta_r + 2\Delta\theta_H \leq \theta_1 \leq \theta_r + \theta_H - 2\Delta\theta_H$ and $\theta_r + \theta_H + \theta_f \leq \theta_1 \leq 360$, roller $r1$ and roller $r2$ both contact the cam profile, and hence the pressure angle can be regarded as zero due to the symmetrical arrangement of the rollers and cam.

3. For $\theta_r + \theta_H - 2\Delta\theta_H \leq \theta_1 \leq \theta_r + \theta_H + \theta_f$, only roller $r2$ contacts the cam profile, and the pressure angle can be derived using Eq. (20).

4. ANALYSIS OF CURVATURE

The principal curvatures of the cam surface must be analyzed in order to prevent the occurrence of singular points and to determine the size of the cam surface. In general, the absolute value of the negative radius of curvature of the cam profile should be greater than the radius of the roller such that the roller can fit into the concave part of the cam profile. As a result, the principal curvatures of the cam profile must be carefully designed. According to basic differential geometry principles, the principal curvatures of the cam profile can be evaluated as follows:

$$K_1, K_2 = H \pm \sqrt{H^2 - K}, \tag{21}$$

where K_1 and K_2 are the principal curvatures, and K and H are defined respectively as

$$K = \frac{LN - M^2}{EG - F^2} \quad \text{and} \quad H = \frac{2FM - EN - GL}{2(EG - F^2)}, \tag{22}$$

where

$$L = {}^0n \bullet \frac{\partial^2\, {}^0S}{\partial^2 u}, \quad M = {}^0n \bullet \frac{\partial^2\, {}^0S}{\partial u \partial \theta_1}, \quad N = {}^0n \bullet \frac{\partial^2\, {}^0S}{\partial^2 \theta_1}, \quad E = \frac{\partial\, {}^0S}{\partial u} \bullet \frac{\partial\, {}^0S}{\partial u}, \quad F = \frac{\partial\, {}^0S}{\partial u} \bullet \frac{\partial\, {}^0S}{\partial \theta_1},$$

and

$$G = \frac{\partial\, {}^0S}{\partial \theta_1} \bullet \frac{\partial\, {}^0S}{\partial \theta_1}$$

(see Eqs. (A1–A6) in Appendix A). Note that the formulas in this appendix correspond to roller $r1$. For the case of roller $r2$, the term e should be replaced with $-e$ and the term $-e$ becomes e.

As shown in Eqs. (21) and (22), the principal curvatures of the cam profile are related to both the follower motion program and the dimensions of the kinematic model for the cam mechanism.

5. IMPLEMENTATION

To validate the design methodology presented in Sections 2–4, a cam mechanism with a translating follower having double rollers was constructed with design parameters of $\theta_r = 100°$, $\theta_H = 70°$, $\theta_f = 130°$, $\theta_L = 60°$, $e = 10$ mm, $R_0 = 40$ mm and $\rho = 8$ mm. The follower displacement was specified by a modified sine motion curve, in which θ_1 is the rotational position of the cam, s_2 is the translation displacement of the follower, and θ_d is the dwell period. In general, a modified sine motion curve starts from a zero position and rises to a total height h over the period of the cam rotation, τ. In the present study, h was assigned a value of 24 mm.

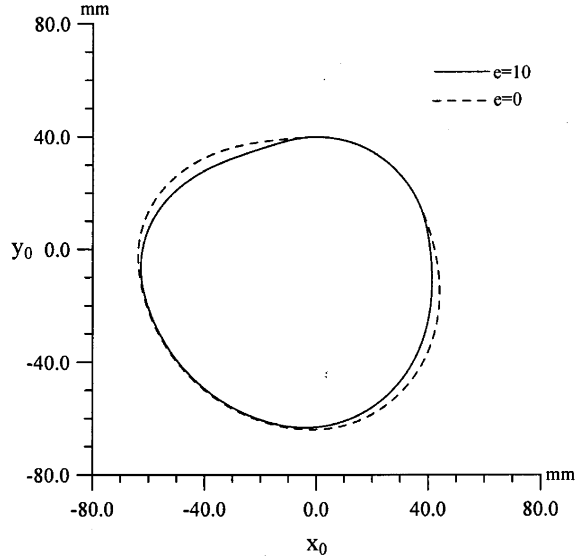

Fig. 4. Simulated profiles of cam given use of symmetric double rollers ($e = 10$) and conventional single roller ($e = 0$).

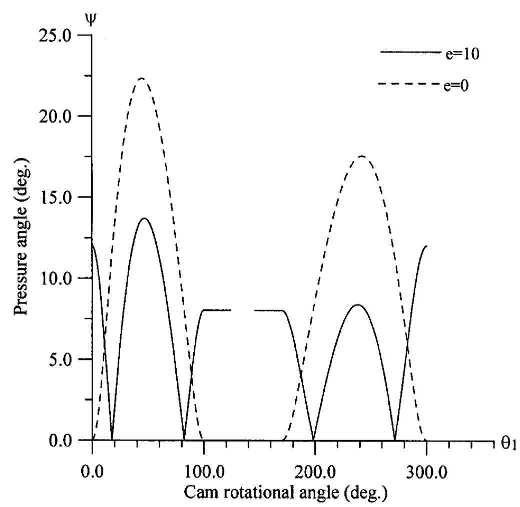

Fig. 5. Variation of pressure angle for two different cam profile designs.

Moreover, the follower displacement, $s2$, associated with the modified sine curve was given as follows:

$$s_2(\theta_1) = \begin{cases} h\left[\frac{\pi}{4+\pi}\,\frac{\theta_1-\theta_d}{\tau} - \frac{1}{4(4+\pi)}S\left(4\pi\frac{\theta_1-\theta_d}{\tau}\right)\right], 0 \leq \theta_1 - \theta_d \leq \frac{\pi}{8} \\ h\left[\frac{2}{4+\pi} + \frac{\pi}{4+\pi}\,\frac{\theta_1-\theta_d}{\tau} - \frac{9}{4(4+\pi)}S\left(\frac{4\pi}{3}\,\frac{\theta_1-\theta_d}{\tau} + \frac{\pi}{3}\right)\right], \frac{\pi}{8} \leq \theta_1 - \theta_d \leq \frac{7\tau}{8} \\ h\left[\frac{4}{4+\pi} + \frac{\pi}{4+\pi}\,\frac{\theta_1-\theta_d}{\tau} - \frac{1}{4(4+\pi)}S\left(4\pi\frac{\theta_1-\theta_d}{\tau}\right)\right], \frac{7\tau}{8} \leq \theta_1 - \theta_d \leq \tau \end{cases} . \quad (23)$$

Figure 4 shows the simulation results obtained for the cam profile given the use of a symmetric double roller ($e = 10$) and common single roller ($e = 0$), respectively. Figure 5 presents the corresponding results for the variation of the pressure angle at the contact point as the cam rotates. It is seen that the symmetric double roller results in a significant reduction in the peak pressure angle. Figure 6 shows the principal

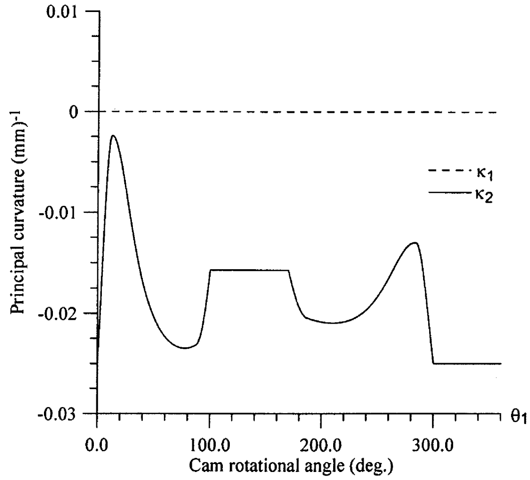

Fig. 6. Variation of principal curvatures of designed cam profile with symmetric double rollers.

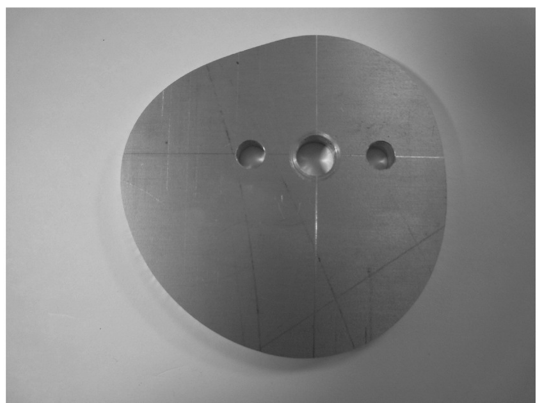

Fig. 7. Photograph of machined cam.

curvatures of the cam profile designed for the double-roller follower. It is noted that one of the principal curvatures (i.e., K_1) is equal to zero since the cam has a 2D profile. The results indicate that the designed cam has a minimum radius of curvature of 40.0 mm. Thus, the roller maintains continuous contact with the cam.

The NC equations required to produce the corresponding cam using a 3-axis machine tool were derived using the methodology described by the present group in [6]. Figure 7 presents a photograph of the machined cam. Note that the three holes lying along the x_0 axis of the cam are the workpiece origin (center hole) and the clamping holes, respectively. It is seen that a good qualitative agreement exists between the machined profile and the designed profile (see Fig. 4). Thus, the feasibility of the proposed methodology is confirmed.

6. CONCLUSIONS

This paper has presented an integrated methodology for the design, analysis and machining of a cam mechanism incorporating a translating follower with a double roller. A kinematic model of the cam mechanism has been derived utilizing the homogeneous coordinate transformation method and conjugate surface theory. In addition, analytical expressions have been derived for the pressure angle and principal curvatures of the cam profile. The NC data required to machine the designed cam have been obtained using the method proposed by the present group in a previous study [6]. It has been shown that the profile of the machined cam is in good agreement with the analytically-derived profile.

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APPENDIX A

In deriving the principal curvatures of the cam profile using Eqs. (21) and (22), the related parameter definitions are as follows:

$$L = 0, \tag{A.1}$$

$$M = 0, \tag{A.2}$$

$$N = -r\left(\frac{d\bar{\theta}}{d\theta_1}+1\right)^2 + eC\bar{\theta} - b_2 S\bar{\theta} - 2C\bar{\theta}\frac{db_2}{d\theta_1} + S\bar{\theta}\frac{d^2b_2}{d\theta_1^2}, \tag{A.3}$$

$$E = 1, \tag{A.4}$$

$$F = 0, \tag{A.5}$$

$$G = r^2\left(\frac{d\bar{\theta}}{d\theta_1}+1\right)^2 + (e)^2 + (b_2)^2 + 2rC(\bar{\theta})\left(\frac{d\bar{\theta}}{d\theta_1}+1\right) + 2rb_2S(\bar{\theta}) + \left(\frac{db_2}{d\theta_1}\right)^2 - 2e\frac{db_2}{d\theta_1}, \tag{A.6}$$

where

$$\frac{d\bar{\theta}}{d\theta_1} = \frac{\left(-e+\frac{db_2}{d\theta_1}\right)\frac{db_2}{d\theta_1} - b_2\left(\frac{d^2b_2}{d\theta_1^2}\right)}{b_2^2 + \left(-e+\frac{db)2}{d\theta_1}\right)^2}.$$