ESTABLISHMENT OF DYNAMIC EQUATIONS FOR DAMPED SYSTEMS BASED ON QUASI-VARIATIONAL PRINCIPLES

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ABSTRACT
In this paper, quasi-variational principles for non-conservative damped systems are studied. A Hamilton-type quasi-variational principle for non-conservative systems in analytical mechanics and a quasi-variational principle of potential energy in non-conservative elastodynamics systems are proposed in simplified forms respectively, by using the direct variational integral method. On the basis of the standard linear solid model for viscoelastic materials, the dynamic equations of exponentially damped systems are established through the proposed quasi-variational principles. A distinction between the internal damping described by exponential damping and the external damping described by viscous one in a vibrating structure is according to different physical mechanisms, which gives some indication of the correct mechanism of damping.

Keywords: quasi-variational principle; non-conservative damped system; non-viscous damping; exponential damping.

ÉTABLISSEMENT D'ÉQUATIONS DYNAMIQUES POUR LES SYSTÈMES BASÉS SUR DES PRINCIPES AMORTIS QUASI-VARIATIONNELS

RÉSUMÉ
Dans cet article sont étudiés, les principes quasi-variationnels pour les systèmes non-conservatifs amortis. En mécanique analytique, un principe quasi-variationnel de type Hamilton pour les systèmes non-conservatifs, et un principe quasi-variationnel d'énergie potentielle dans les systèmes non- conservatifs élastodynamiques sont proposés respectivement dans des formes simplifiées, en utilisant la méthode variationnelle intégrale directe. Sur la base du modèle solide linéaire standard pour les matériaux viscoélastiques, les équations dynamiques des systèmes amortis de façon exponentielle sont établies par les principes quasi-variationnels proposés. Une distinction entre l'amortissement interne décrit par l'amortissement exponentiel et l'amortissement externe décrit par une vibration visqueuse dans la structure se fait selon différents mécanismes physiques, lesquels donnent certaines indications sur le mécanisme d'amortissement juste.

Mots-clés : principe quasi-variationnel; système amorti non-conservatif; amortissement non-visqueux; amortissement exponentiel.

1. INTRODUCTION

Damping is one of the important dynamic characteristics in a vibrating system [1, 2]. In engineering practice, the most popular method is to assume viscous damping for the reason of mathematical simplicity. A further assumption of viscous damping is first introduced by Rayleigh [3], so-called 'proportional damping' or 'classical damping', whose damping coefficient matrix is the linear combination of the system's mass and stiffness matrices. For a proportionally damped system, the equations of motion can be simply decoupled into a set of single degree of freedom (DOF) systems using real normal modes as in the undamped case. Caughey [4] derived the general conditions in the form of proportional damping matrix and presented a series expression for the damping matrix in terms of mass and stiffness matrices, in which the Rayleigh damping is a special case of this general expression. Later Adhikari [5, 6] presented a further generalization of proportional damping in which the damping matrix can be expressed in terms of smooth continuous functions involving specially arranged mass and stiffness matrices.

Engineering structures generally exhibit nonviscous damping behaviour. Recently, there has been an increase in interest on other damping models, in which the dissipative forces depend on any quantity other than the instantaneous generalized velocities, like hysteretic models [7–11], fractional derivative models [12, 13], and viscoelastic or nonviscous models [14–16], with an aim to represent damping energy dissipation in a more suitable manner. Mathematically, any model that makes the energy dissipation functional non-negative is a possible candidate for a nonviscous damping model. A more popular nonviscous damping is the convolution integral model, in which the damping forces are assumed to depend on the past history of velocities via convolution integrals over kernel functions. There have been exhaustive studies on the analysis of such damping systems, for instance, eigenvalue problems [17–19], dynamic analysis [20–23], damping identification [24] and so on. Although a wide variety of mathematical expressions could be used for the kernel functions, there appears to be a lack of a full understanding of physical meaning. Therefore, the physical basis for an exponential damping model, a particular type of nonviscous damping which has been used by many authors, was hopefully established through theoretical derivations based on variational principle.

At present, the applications of generalized variational principles in deformable body mechanics have gained great success in the field of finite element method (FEM) and other approximate computational methods [25–29]. Meanwhile, these principles have been extended to the analytical mechanics by many authors [30–32]. However, the traditional variational principles are only suitable for conservative systems. It is difficult to establish the variational principles for a non-conservative system. Leipholz [33, 34] proposed the direct methods in the calculus of variations to establish the Hamilton-type quasi-variational principle. After that, researches began to focus and study on the variational principles for non-conservative damped systems in the field of elastodynamics [35, 36] and analytical dynamics [37–39], and establish various types of quasi-variational principles under various application backgrounds.

In this paper, further studies on the quasi-variational principles for non-conservative damped systems are conducted on the basis of the previously mentioned work. First, a Hamilton-type quasi-variational principle for non-conservative systems in analytical mechanics and a quasi-variational principle of potential energy in non-conservative elastodynamics systems are established in simplified forms, by using the direct variational integral method, respectively. Second, based on the assumption of the standard linear solid model for viscoelastic materials, the dynamic equations of exponentially damped systems are presented through the theoretical derivation.

2. HAMILTON-TYPE QUASI-VARIATIONAL PRINCIPLES IN ANALYTICAL MECHANICS

2.1. Basic Equations of Analytical Mechanics

In analytical mechanics, the equation of motion of a non-conservative viscously damped system can be expressed as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k \tag{1}$$

where q_k, for $k = 1, 2, 3 \ldots n$, are the generalized coordinates, T is the kinetic energy, and Q_k are the external forces, which include conservative and non-conservative forces, can be written as

$$Q_k = -\frac{\partial U}{\partial q_k} + Q'_k - \frac{\partial \Phi}{\partial \dot{q}_k} \tag{2}$$

where U and Φ are the potential energy and damping dissipation energy. The first term in Eq. (2) denotes the conservative external forces. Q'_k are the non-conservative forces excluding the damping force. It is assumed that the damping force in viscously damped systems is proportional to the velocity. In this case, damping dissipation energy takes the form as

$$\Phi = \sum_{k=1}^{n} \frac{1}{2} c \dot{q}_k^2 \tag{3}$$

where c is the viscous damping coefficient.

Multiplying both sides of Eq. (1) by the corresponding variation δq_k, summing over all k, and then integrating with respect to time, one obtains

$$\int_{t_0}^{t_1} \sum_{k=1}^{n} \left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} \right) \delta q_k dt = \int_{t_0}^{t_1} \sum_{k=1}^{n} \left(-\frac{\partial U}{\partial q_k} + Q'_k - \frac{\partial \Phi}{\partial \dot{q}_k} \right) \delta q_k dt \tag{4}$$

It is commonly assumed that $\delta q_k(t_0) = \delta q_k(t_1) \equiv 0$ holds true. Integrating the first term in the integrand of the left-hand side of Eq. (4) by part gives

$$\int_{t_0}^{t_1} \sum_{k=1}^{n} \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} \delta q_k dt = \sum_{k=1}^{n} \frac{\partial T}{\partial \dot{q}_k} \delta q_k \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \sum_{k=1}^{n} \frac{\partial T}{\partial \dot{q}_k} \delta \dot{q}_k dt = -\int_{t_0}^{t_1} \sum_{k=1}^{n} \frac{\partial T}{\partial \dot{q}_k} \delta \dot{q}_k dt \tag{5}$$

Substituting Eq. (5) into Eq. (4), Eq. (4) can be further rearranged and reads

$$\delta \int_{t_0}^{t_1} [T(t, q_k, \dot{q}_k) - U(t, q_k)] dt + \int_{t_0}^{t_1} \sum_{k=1}^{n} Q'_k \delta q_k dt - \int_{t_0}^{t_1} \sum_{k=1}^{n} \frac{\partial \Phi}{\partial \dot{q}_k} \delta q_k dt = 0 \tag{6}$$

For the sake of brevity, it can be written as

$$\delta H + \delta Q_H - \delta \Phi_H = 0 \tag{7}$$

where $\delta H = 0$, the first term of Eq. (7), is known as Hamilton's principle. Since Eq. (7) is derived from the fundamental principle by using the integral formula, it shall be called Hamilton-type quasi-variational principle.

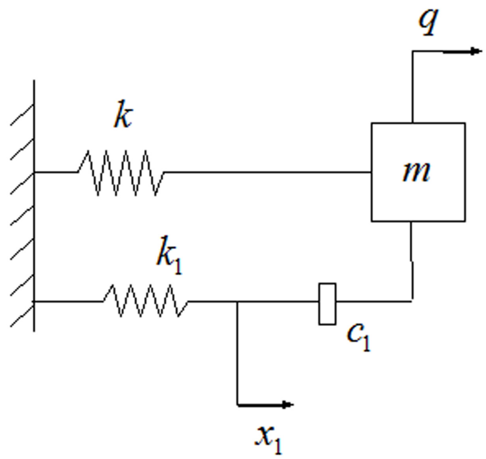

Fig. 1. Single DOF vibration system with exponential damping.

2.2. Establishment of Dynamic Equations for Exponentially Damped Systems

This section, the application of variational principles in exponentially damped systems is investigated. Firstly, a single DOF system with exponential damping is considered, as shown in Fig. 1.

The mass and stiffness of the system are m and k, respectively. Note that k_1 is the so-called damping-stiffness, which is related to the damping mechanism. c_1 is the damping coefficient of a viscous damper. Suppose that q is the generalized coordinate of particle motion and x_1 is the given coordinate, corresponding to the damp-stiffness. $f(t)$ is the non-conservative time-varying force.

According to the principle of viscous damping energy dissipation, the dissipative function of this system can be written as

$$\Phi = \frac{1}{2}c_1(\dot{q}-\dot{x}_1)^2 \tag{8}$$

On the basis of the quasi-variation principle presented above, the kinetic energy and the potential energy of the system can be first obtained as

$$T = \frac{1}{2}m\dot{q}^2, \quad U = \frac{1}{2}kq^2 + \frac{1}{2}k_1x_1^2 \tag{9}$$

Applying integration by parts and defining the δq_k is zero at boundary of time domain, one obtains

$$\delta\int_{t_0}^{t_1} T\,dt = \int_{t_0}^{t_1} m\dot{q}\delta\dot{q}dt = -\int_{t_0}^{t_1} m\ddot{q}\delta q dt \tag{10}$$

$$\delta\int_{t_0}^{t_1} U\,dt = \int_{t_0}^{t_1} kq\delta q + k_1x_1\delta x_1 dt \tag{11}$$

$$\int_{t_0}^{t_1}\sum_{k=1}^{n}\frac{\partial\Phi}{\partial\dot{q}_k}\delta q_k dt = \int_{t_0}^{t_1}[c_1(\dot{q}-\dot{x}_1)\delta q - c_1(\dot{q}-\dot{x}_1)\delta x_1]dt \tag{12}$$

$$\delta Q_H = \int_{t_0}^{t_1} f\delta q dt \tag{13}$$

Substituting Eqs. (10)–(13) into Eq. (7) yields finally

$$\delta H + \delta Q_H - \delta\Phi_H = \int_{t_0}^{t_1}\left\{\delta q[-m\ddot{q} - kq - c_1(\dot{q}-\dot{x}_1) + f] + \delta x_1[-k_1x_1 + c_1(\dot{q}-\dot{x}_1)]\right\}dt = 0 \tag{14}$$

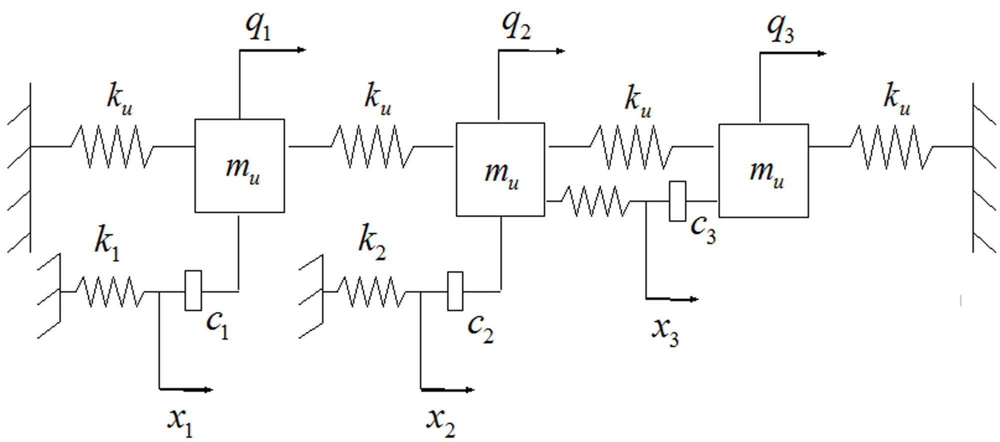

Fig. 2. Three DOFs vibration system with exponential damping.

δq_k, $\delta x_1 = 0$ at fixed end points $t = t_0$ and $t = t_1$ are considered. Hence

$$\begin{cases} m\ddot{q} + kq + c_1(\dot{q} - \dot{x}_1) = f \\ k_1 x_1 = c_1(\dot{q} - \dot{x}_1) \end{cases} \tag{15}$$

The solution of the second equation of Eq. (15) is obtained:

$$x_1 = \int_0^t e^{(-k_1/c_1)(t-\tau)} \dot{q}(\tau) d\tau \tag{16}$$

Defining k_1 by using relaxing factor μ_1, hence, the dynamic equation of the system can be written as

$$m\ddot{q} + c_1 \int_0^t \mu_1 e^{-\mu_1(t-\tau)} \dot{q}(\tau) d\tau + kq = f \tag{17}$$

Equation (17) is the dynamic equation of the single DOF vibration system with exponential damping.

A three DOFs system shown in Fig. 2 is considered in the following section. In this system, the damping model is considered as an exponential one. It is assumed that the damp-stiffness of the dampers are k_1, k_2 and k_3, respectively; generalized coordinates of particle motion are q_k, for $k = 1,2,3$; coordinates of the damp-stiffness are x_k, for $k = 1,2,3$.

Similar to the method for the single DOF exponentially damped system depicted above, based on Hamilton variation principle, we can obtain a set of six equations as

$$\begin{cases} -m_u\ddot{q}_1 - k_u q_1 + k_u(q_2 - q_1) - c_1(\dot{q}_1 - \dot{x}_1) + f_1 = 0 \\ -m_u\ddot{q}_2 - k_u(q_2 - q_1) + k_u(q_3 - q_2) + k_3(x_3 - q_2) - c_2(\dot{q}_2 - \dot{x}_2) + f_2 = 0 \\ -m_u\ddot{q}_3 - k_u(q_3 - q_2) - k_u q_3 - c_3(\dot{q}_3 - \dot{x}_3) + f_3 = 0 \\ -k_1 x_1 + c_1(\dot{q}_1 - \dot{x}_1) = 0 \\ -k_2 x_2 + c_2(\dot{q}_2 - \dot{x}_2) = 0 \\ -k_3(x_3 - q_2) + c_3(\dot{q}_3 - \dot{x}_3) = 0 \end{cases} \tag{18}$$

The analytical solution of the last three equations are shown as follows:

$$x_1 = \int_0^t e^{-(k_1/c_1)(t-\tau)} \dot{q}_1(\tau) d\tau, \quad x_2 = \int_0^t e^{-(k_2/c_2)(t-\tau)} \dot{q}_2(\tau) d\tau, \quad x_3 - q_2 = \int_0^t e^{-(k_3/c_3)(\tau-t)} [\dot{q}_3(\tau) - \dot{q}_2(\tau)] d\tau \tag{19}$$

Defining the relaxing factor $\mu_k = k_k/c_k$, for $k = 1,2,3$ and substituting Eq. (19) into Eq. (18) yields

$$\begin{cases} m_u\ddot{q}_1 + 2k_u q_1 - k_u q_2 + c_1\mu_1 \int_0^t e^{-\mu_1(t-\tau)}\dot{q}_1(\tau)d\tau = f_1 \\ m_u\ddot{q}_2 - k_u q_1 + 2q_2 k_u - k_u q_3 + c_2\mu_2 \int_0^t e^{-\mu_2(t-\tau)}\dot{q}_2(\tau)d\tau - c_3\mu_3 \int_0^t e^{-\mu_3(t-\tau)}[\dot{q}_3(\tau) - \dot{q}_2(\tau)]d\tau = f_2 \\ k_u q_2 + 2k_u q_3 + c_3\mu_3 \int_0^t e^{-\mu_3(t-\tau)}[\dot{q}_3(\tau) - \dot{q}_2(\tau)]d\tau = f_3 \end{cases} \tag{20}$$

which can be transformed into a matrix form:

$$\mathbf{M}\ddot{\mathbf{q}} + \sum_{k=1}^{3} \int_0^t \mathbf{C}_k \mu_k e^{-\mu_k(t-\tau)}\dot{\mathbf{q}}(\tau)d\tau + \mathbf{K}\mathbf{q} = \mathbf{f} \tag{21}$$

where

$$\mathbf{M} = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_u & 0 \\ 0 & 0 & m_u \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2k_u & -k_u & 0 \\ -k_u & 2k_u & -k_u \\ 0 & -k_u & 2k_u \end{bmatrix} \tag{22}$$

$$\mathbf{C}_1 = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \tag{23}$$

Equation (21) is the dynamic equation of the three DOF exponentially damped system.

3. QUASI-VARIATIONAL PRINCIPLE OF POTENTIAL ENERGY IN LINEAR ELASTODYNAMICS

3.1. Basic Equations in Linear Elastodynamics

The basic equations of linear elastodynamics for a viscously damped system can be written as follows:

Equilibrium equation:

$$\sigma_{ij,j} + F_i - \rho\ddot{u}_i - c\dot{u}_i = 0 \tag{24}$$

Geometric equation

$$\varepsilon_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i}) = 0 \tag{25}$$

Constitutive equation

$$\sigma_{ij} = \sigma_{ij}^E + \sigma_{ij}^D = \left(\lambda + \lambda'\frac{\partial}{\partial t}\right)\delta_{ij}e + 2\left(\mu + \mu'\frac{\partial}{\partial t}\right)\varepsilon_{ij} \tag{26}$$

Boundary conditions

$$\sigma_{ij}n_j - P_i = 0, \quad u_i - \bar{u}_i = 0, \quad \text{on } S_\sigma \tag{27}$$

Initial conditions

$$u_i|_{t=0} = d_i, \quad \dot{u}_i|_{t=0} = v_i, \quad \text{in } V \tag{28}$$

where ρ is the mass density; u_i is the displacement; symbols $\dot{()}$ and $\ddot{()}$ denote the first and second derivatives with respect to time; F_i and P_i are the volume force and surface force, respectively; σ_{ij} and ε_{ij} are the stress and the strain, respectively; λ and μ are the Lame's constant; λ' and μ' are the coefficients of viscous damping in the material constitutive relation; σ_{ij}^E is the elastic stress only proportional to elastic strain; σ_{ij}^D is the damping stress which is proportional to strain rate; c is the external viscous damping coefficient.

Multiplying both Eqs. (24) and (27) by corresponding δu_i, integrating them with respect to volume and time, and adding, we obtain

$$\int_{t_0}^{t_1}\int_V(\sigma_{ij,j}+F_i-\rho\ddot{u}-c\dot{u}_i)\delta u_i dVdt-\int_{t_0}^{t_1}\int_{S\sigma}-P_i)\delta u_i dSdt=0 \tag{29}$$

Applying divergence theorem to Eq. (29) yields

$$\int_V\sigma_{ij,j}\delta u_i dV=\int_V(\sigma_{ij}\delta u_i)_{,j}dV-\int_V\sigma_{ij}\delta u_{i,j}dV=\int_{S\sigma+Su}\sigma_{ij}n_j\delta u_i dS-\int_V\sigma_{ij}\delta u_{i,j}dV \tag{30}$$

Since δu_i is arbitrary in Su, a given displacement boundary condition thus yields

$$\int_V\sigma_{ij,j}\delta u_i dV=\int_{S\sigma}\sigma_{ij}n_j\delta u_i dS-\int_V\sigma_{ij}\delta u_{i,j}dV \tag{31}$$

and because of

$$\delta u_{i,j}=\delta\left[\frac{1}{2}(u_{i,j}+u_{j,i})+\frac{1}{2}(u_{i,j}-u_{j,i})\right]=\delta(\varepsilon_{ij}+\omega_{ij})=\delta\varepsilon_{ij} \tag{32}$$

where ω_{ij} is rotational strain, can be zero if not considered. Substituting Eq. (32) into Eq. (31) yields

$$\int_V\sigma_{ij,j}\delta u_i dV=\int_{S\sigma}\sigma_{ij}n_j\delta u_i dS-\int_V\sigma_{ij}\delta\varepsilon_{ij}dV \tag{33}$$

The elastic strain energy and strain-rate dependent viscous damping dissipation function of unit volume can be expressed as

$$U_0=\frac{1}{2}\sigma_{Eij}\varepsilon_{ij},\quad \Phi_{02}=\frac{1}{2}\sigma_{ij}^D\dot{\varepsilon}_{ij} \tag{34}$$

It can be improved so that

$$\delta U_0=\sigma_{ij}^E\delta\varepsilon_{ij},\quad \sigma_{ij}^D\delta\varepsilon_{ij}=\frac{\partial\Phi_{02}}{\partial\dot{\varepsilon}_{ij}}\delta\varepsilon_{ij} \tag{35}$$

Then

$$\sigma_{ij}\delta\varepsilon_{ij}=\sigma_{ij}^E\delta\varepsilon_{ij}+\sigma_{ij}^D\delta\varepsilon_{ij}=\delta U_0+\frac{\partial\Phi_{02}}{\partial\dot{\varepsilon}_{ij}}\delta\varepsilon_{ij} \tag{36}$$

Substitute Eq. (36) into Eq. (33), it can be expressed as

$$\int_V\sigma_{ij,j}\delta u_i dV=\int_{S\sigma}\sigma_{ij}n_j\delta u_i dS-\int_V\delta U_0 dV-\int_V\frac{\partial\Phi_{02}}{\partial\dot{\varepsilon}_{ij}}\delta\varepsilon_{ij}dV \tag{37}$$

Due to the arbitrariness of in the given initial boundary condition, $\delta u_i|_{t_0}^{t_1}=0$, applying integration by parts, we can obtain

$$\int_{t_0}^{t_1}\rho\ddot{u}_i\delta u_i dt=0-\int_{t_0}^{t_1}\rho\dot{u}_i\delta\dot{u}_i dt=-\int_{t_0}^{t_1}\delta T_0 dt \tag{38}$$

where $T_0=\rho\dot{u}_i^2/2$ is the kinetic energy of unit volume.

The unit energy dissipation function of external damping can be expressed as

$$\Phi_{01}=\frac{1}{2}c\dot{u}_i^2\quad\text{and}\quad c\dot{u}_i\delta u_i=\frac{\partial\Phi_{01}}{\partial\dot{u}_i}\delta u_i \tag{39}$$

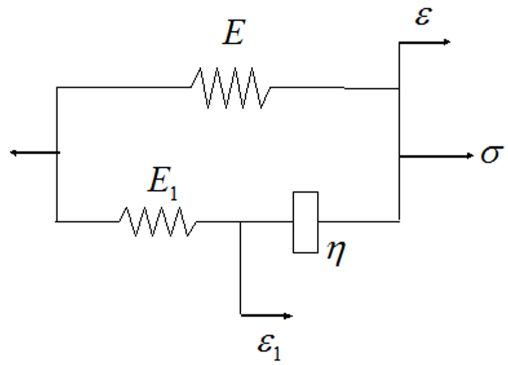

Fig. 3. Constitutive relation of exponential damping model.

Both the volume force and the surface force contain two parts: the potential force and the non-potential force,

$$F_i\delta u_i = -\frac{\delta U_{0F}}{\partial u_i}\delta u_i + F_i'\delta u_i = -\delta U_{0F} \quad \text{and} \quad P_i\delta u_i = -\frac{\delta U_{0P}}{\partial u_i}\delta u_i + P_i'\delta u_i = -\delta U_{0P} + P_i'\delta u_i \tag{40}$$

Consequently, according to Eqs. (37)–(40), Eq. (29) can be transformed into

$$\begin{aligned} &\delta\int_{t_0}^{t_1}(T-U)dt + \int_{t_0}^{t_1}\int_V F_i'\delta u_i dV dt + \int_{t_0}^{t_1}\int_{S\sigma} P_i'\delta u_i dS dt \\ &-\int_{t_0}^{t_1}\int_V \frac{\partial\Phi_{02}}{\partial\dot{\varepsilon}_{ij}}\delta\varepsilon_{ij} dV dt - \int_{t_0}^{t_1}\int_V \frac{\partial\Phi_{01}}{\partial\dot{u}_i}\delta u_i dV dt = 0 \end{aligned} \tag{41}$$

where

$$\delta(T-U) = \int_V(\delta T_0 - \delta U_0 - \delta U_{0F})dV - \int_{t_0}^{t_1}\int_{S\sigma}\delta U_{0P} dS dt \tag{42}$$

Equation (41) can be written in abbreviated form as

$$\delta H + \delta Q_H - \delta\Phi_H = 0 \tag{43}$$

Equation (43) is the quasi-variational principle of potential energy for linear elastodynamics.

3.2. Establishment of Motion Equation for a Transversely Vibrating Beam with Exponential Damping

In this section, the transverse bending vibration problem of an exponentially damped beam is considered. The exponential damping model adopted in the material constitutive relation can be illustrated in Fig. 3.

It is assumed that and are elastic modulus and strain, respectively. Their corresponding generalized coordinate is $w(x,t)$; E_1 and ε_1 are the damping modulus and strain, respectively. Their corresponding generalized coordinate is assumed to be $w_1(x,t)$. For this case, the kinetic energy and the potential energy of the beam-like system can be expressed as follows:

Kinetic energy

$$T = \int_V \frac{1}{2}\rho\dot{w}^2 dV = \int_0^t \frac{1}{2}\rho A\dot{w}^2 dx \tag{44}$$

Potential energy

$$U=\int_V \frac{1}{2}E\varepsilon^2 dV+\int_V \frac{1}{2}E_1\varepsilon_1^2 dV=\int_0^l \frac{1}{2}EIw''^2dx+\int_0^l \frac{1}{2}E_1Iw_1''^2dx \tag{45}$$

where $I(x)=\int_S y^2dS$.

Furthermore, two kinds of damping dissipation function can be written as

$$\Phi_{01}=\frac{1}{2}c\dot{w}^2,\quad \Phi_{02}=\frac{1}{2}\eta(\dot{\varepsilon}-\dot{\varepsilon}_1)^2 \tag{46}$$

Integrating the kinetic energy and the strain potential energy with respect to time and applying the corresponding variation results in

$$\delta\int_{t_0}^{t_1} Tdt=\int_{t_0}^{t_1}\rho A\dot{w}\delta\dot{w}dxdt=-\int_{t_0}^{t_1}\int_0^l \rho A\ddot{w}\delta wdxdt \tag{47}$$

in which one integration by parts is used considering that $\delta w=0$ in the time boundary.

$$\delta\int_{t_0}^{t_1} Udt=\int_{t_0}^{t_1}\int_0^l EIw''\delta w''dxdt+\int_{t_0}^{t_1}\int_0^l E_1Iw''\delta w_1''dxdt=\int_{t_0}^{t_1}\int_0^l (EIw'')''\delta wdxdt+\int_{t_0}^{t_1}(EIw'')''\delta w_1dxdt \tag{48}$$

is obtained by means of two integration by parts considering that w, w_1 and w_1' are all zero on the boundary; then

$$\delta H=-\int_{t_0}^{t_1}\int_0^l\{[\rho A\ddot{w}+(EIw'')'']\delta w+(EIw'')''\delta w_1\}dxdt \tag{49}$$

For the damping dissipation functions, we have

$$\int_{t_0}^{t_1}\int_V \frac{\partial\Phi_{01}}{\partial\dot{w}}\delta wdVdt=\int_{t_0}^{t_1}\int_0^l cA\dot{w}\delta wdxdt \tag{50}$$

$$\begin{aligned}\int_{t_0}^{t_1}\int_V\left(\frac{\partial\Phi_{02}}{\partial\dot{\varepsilon}}\delta\varepsilon+\frac{\partial\Phi_{02}}{\partial\dot{\varepsilon}_1}\delta\varepsilon_1\right)dVdt&=\int_{t_0}^{t_1}\int_V \eta(\dot{\varepsilon}-\dot{\varepsilon}_1)\delta\varepsilon-\eta(\dot{\varepsilon}-\dot{\varepsilon}_1)\delta\varepsilon_1 dVdt\\&=\int_{t_0}^{t_1}\int_0^l[\eta I(\dot{w}''-\dot{w}_1'')\delta w''-\eta](\dot{w}''-\dot{w}_1'')\delta w_1'']dxdt\\&=\int_{t_0}^{t_1}\int_0^l\left\{[\eta I(\dot{w}''-\dot{w}_1'')]^n\delta w-[\eta I(\dot{w}''-\dot{w}_1'')]^n\delta w_1\right\}dxdt\end{aligned} \tag{51}$$

in which two integration by parts is also used.

Finally, considering the transverse distributed force , we can also write

$$\delta Q_H=\int_{t_0}^{t_1}\int_{S\sigma}q(x,t)\delta wdSdt=\int_{t_0}^{t_1}\int_0^l q(x,t)\delta wdxdt \tag{52}$$

Substituting Eqs. (49)–(52) into Eq. (43), we can obtain the following equations:

$$\begin{cases}\rho A\ddot{w}+(EIw'')''+[\eta I(\dot{w}''-\dot{w}_1'']''+cA\dot{w}=q\\ [\eta I(\dot{w}''-\dot{w}_1'')]''=(E_1Iw_1'')''\end{cases} \tag{53}$$

In this case, we need to introduce a relaxing factor $\mu=E_1/\eta$.

The analytical solution of the second equation of Eq. (53) can be written as

$$(E_1 I w_1'')'' = \int_0^l \mu e^{-\mu(t-\tau)} [\eta I \dot{w}''(\tau)]'' d\tau \tag{54}$$

Introducing the proportional damping coefficients α_0, α_1, $c = \alpha_0 \rho$ and $\eta = \alpha_1 E$, Eq. (53) can be transformed into

$$\rho A(\ddot{w} + \alpha_0 \dot{w}) + \left\{ EI \left[w'' + \alpha_1 \int_0^l \mu e^{-\mu(t-\tau)} \dot{w}''(\tau) d\tau \right] \right\}'' = q \tag{55}$$

which is the motion equation for transverse vibration of exponentially damped beam.

4. DISCUSSIONS AND CONCLUSIONS

Non-conservative damped systems were considered in this paper. Based on the direct variational methods, the Hamilton-type quasi-variational principle in analytical mechanics was first established, according to the governing equations of initial value problems. Then the quasi-variational principle of potential energy in elastodynamics was established as well, according to the basic equations of initial- boundary-value problems. The equivalence property between the dynamic equations of non-conservative systems and their corresponding variational forms was improved. The establishment of dynamic equations of damped systems was studied on the basis of the two succinct kinds of quasi-variation principle proposed in this paper. With the studies of the classical viscous damping model, the validity of the proposed theoretical method was proved. Finally, the establishment of dynamic equations for exponentially damped systems was further discussed as well.

It could be concluded that it is necessary to make a distinction between the external damping and the internal damping of a vibrating structure according to different physical mechanisms. Although the external damping which is proportional to mass and the internal damping which is proportional to stiffness were considered in the traditional Rayleigh proportional damping theory, they were represented by the viscous one without any distinctions. The external damping originated from the external fluid in contact with the structure (such as buildings or aircrafts in the air, and bridges or ships in the water), may be similar to viscous property and can be represented by the classic viscous damping model. While the internal damping originated from the solid materials of a structure, may exist nonviscous damping properties and can be represented by the exponential damping model like the motion equation of transversely vibrating beam described by Eq. (55). Among the variety of nonviscous damping models, the exponential damping model presented in the paper may be the simplest and the most reasonable one to describe the damping properties for solid materials.

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